Check up -problems

(1) Show that a bounded monotone sequence of real numbers is converging.

(2) Calculate the eigenvalues of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(3) Formulate an important result from your undergraduate analysis course.

(4) Formulate an important result from your undergraduate linear algebra course.

Transition course -hw1

Due date: Wednesday, September 8

(1) Show that a bounded, monotone sequence is convergent.

(2) Give an ε-δ proof for

$$\lim_n \left|\frac{5n+n^2}{n^2-5n}\right| \;=\; 1\;.$$

(3) Let $x>0$ show that

$$\lim_n x^{\frac{1}{n}} \;=\; 1\;.$$

(Hint: you may use the continuous function $f(x) = \ln x$ or Bernoulli's inequality.)

(4) Show that the vectors $x_1, ..., x_n \in \mathbb{R}^n$ given by

$$x_j \;=\; (1,1,..., \underbrace{1}_{j\text{-th position}}, 0,0,...,0)$$

are linearly independent.

(5) Show that $A = \{x \in \mathbb{R} \;:\; \exists_{k\in\mathbb{N}} 2k \leq x \leq 2k+1\}$ is closed.

(6) Use the ϵ-δ criterion to show that $f:[0,\infty)\to[0,\infty)$ given by $f(x)=\sqrt{x}$ is continuous.

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Transition course -hw2

Due date: Monday, September 13

(1) Let $1 \le p < \infty$. Show that $\ell_p = \{(x_n) : \left(\sum_n |x_n|^p\right)^{\frac{1}{p}}$ is a complete metric spaces with respect to

$$d_p(x,y) \;=\; \left(\sum_n |x_n - y_n|^p\right)^{\frac{1}{p}} .$$

(Hint: Use $d_p(x,y) \;=\; \lim_m \left(\sum_{n\le m} |x_n - y_n|^p\right)^{\frac{1}{p}}$ in the proof of the triangle inequality.)

(2) Use the result from the lecture for a short proof of the fact

$$\ell_\infty = \{(x_n) : \sup_n |x_n| < \infty\}$$

is complete with respect to $d(x,y) = \sup_n |x_n - y_n|$.

(3) Show that $x_n = \sum_{j\le n} p^j$ is a Cauchy sequence in $(\mathbb{Z}, dd_p)$. Show that there is no limit $x \in \mathbb{Z}$. (Hint use the unique decomposition $x = \sum a_k p^k$, $a_k \in \{0,...,p-1\}$ for positive integers.)

(4) A function $f : X \to Y$ is uniformly continuous of

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x,y\in X}(d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon) .$$

Show that $f : (0,1) \to \mathbb{R}$, $f(x) = 1/x$ is not uniformly continuous.

(5) The space $C_b(X, \mathbb{R})$ is also a vector space over $\mathbb{R}$. Find n-linearly independent elements for
   (a) $X = \{1,...,n\}$ with the discrete metric.
   (b) $X = [0,1]$ with the usual metric. (Hint: polynomials?)

Transition-hw3

Due date: September 20.

(1) (a) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Show that f is Lipschitz if and only if
$$\sup_x |f'(x)|$$
is finite.
(b) Determine the maximal $b > 0$ such that $f(x) = x^2 - x$ has Lipschitz constant ≤ 1 on $[0, b]$.

(2) Let X be a metric space, $0 < c < 1$ and $f : X \to X$ such that
$$d(f(x), f(y)) \leq cd(x, y) .$$
Let $x_0 \in X$ and define inductively $x_{n+1} = f(x_n)$. Show that (x_n) is Cauchy. Study the function $f(x) = 1 - x$ on $[0, 1]$ and show that this does not work for $c = 1$.

(3) Let X be the completion of $(\mathbb{Z}, dd_p)$ and $y \in \mathbb{Z}$.
(a) Show that exists a continuous map $f : X \to X$ such that $f(x) = x + y$ for all $x \in \mathbb{Z}$.
(b) Show that there exists continuous map $add : X \times X \to X$ satisfying $add(x, y) = x + y$ for all $x, y \in \mathbb{Z}$. (Here the distance on $X \times X$ is given by
$$d((x_1, x_2), (y_1, y_2)) \;=\; d(x_1, y_1) + d(x_2, y_2) \text{ .)}$$
(c) What can you say about multiplication? What structure do you expect for X.

(4) (a) Let X be a metric space and $f : X \to Y$ be continuous. Show that $f(K)$ is compact for all $K \subset X$ compact.
(b) Let X be compact metric space and $f : X \to Y$ be bijective continuous map. Show that f^{-1} is continuous.

Transition course -hw5

Due date: Monday, October 4

(1) On $X = \{-1, 1\}^{\mathbb{N}} = \{(\varepsilon_1, \varepsilon_2, \cdots) \; : \; \varepsilon_i = \pm 1\}$ we define the metric

$$d((\varepsilon_i), (\delta_i)) \; = \; \sum_{i=1}^{\infty} 2^{-i} |\varepsilon_i - \delta_i| \; .$$

Show that (X, d) is a compact metric spaces. Show that

$$f((\varepsilon_i)) \; = \; \sum_i \alpha_i \varepsilon_i$$

is continuous if and only if $\sum_i |\alpha_i| < \infty$.

(2) We consider the space $X = C[0, 1]$ and

$$F \; = \; \{t \mapsto \sum_{k=0}^{n-1} a_k t^k \; : \; \sum_{k=0}^{n-1} |a_k| \; \leq \; 1\}$$

Show that F is compact by finding a compact set $C \subset \mathbb{R}^n$ and a continuous function $g : C \to C[0, 1]$ such that $F = f(C)$.

Transition course -hw5

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Show that F is compact by finding a compact set $C \subset \mathbb{R}^n$ and a continuous function $g : C \to C[0, 1]$ such that $F = f(C)$.

Transition-hw7

Due date: November 1

(1) Calculate the Jordan normal form and appropriate bases of the following matrices.

i) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

ii) $B = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$

(2) Let us recall that $\det : (F^n)^n \to F$ is a map which satisfies the following conditions

i) $\det(v_1 + \lambda w_1, v_2, ..., v_n) = \det(v_1, ..., v_n) + \lambda \det(w_1, ..., v_n)$

ii) $\det(v_1, ..., v_{j-1}, v_j, v_{j+1}., v_n) \;=\; (-1)\det(v_j, ..., v_{j-1}, v_1, v_{j+1}., v_n)$,

iii) $\det(e_1, ..., e_n) = 1$

for all $w_1 \in F^n$, $\lambda \in F$, $v_1, ...v_n \in F^n$. Here $e_1, ..., e_n$ are the standard unit vectors. For a matrix $A = [a_{ij}]$ we define the column vectors $v_j = (a_{ij})_{i=1,..,n}$ and

$$\det(A) \;=\; \det(v_1, ..., v_n) \,.$$

In the following you may use that there is only one map $\det : (F^n)^n = \mathcal{M}_n \to F$ satisfying the conditions $i) \to iii)$.

(a) Show that $\det(AB) \;=\; \det(A)\det(B)$.

(b) For a permuation $\pi : \{1, ..., n\}$, we define a linear map $T_\pi(e_i) \;=\; e_{\pi(i)}$. Let A_π be the cooresponding matrix. We denote the group of permutation by S_n. Show that $\varepsilon : S_n \to F$ defined by

$$\varepsilon(\pi) \;=\; \det(A_\pi)$$

is a group homomorphism.

(c) Let $i \neq j$. Show that for a cycle (ij) (which interchanges i and j) we have

$$\varepsilon((ij)) \;=\; -1\,.$$

(d) Show thet every permutation π may be written as a product of cycles (hint: use induction) and that for $\pi = (i_1 j_1) \cdots (i_m j_m)$ we have

$$\varepsilon(\pi) \;=\; (-1)^m\,.$$

(One can actually show that every permutation is a product of neighbouring cycles.)

Transition-hw8

Due date: November 8

(1) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Show that the sequence of polynomials

$$p_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!} A^k$$

converges pointwise in $\mathbb{R}^9$ and calculate the limit. Also find the JNF for A.

(2) Do the same as in 1) but now for $A = \begin{pmatrix} 4 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ and the series for $\cos(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}$. Also compute the JNF.

(3) A liner map $T : V \to V$ is called nilpotent, if there exists $m \in \mathbb{N}$ such that $T^m = 0$. Let A be an upper diagonal matrix and T_A the induced linear map on $\mathbb{C}^n$. A is called nilpotent if T_A is nilpotent. Characterize nilpotent upper diagonal matrices. Find a nilpotent 2×2 matrix with non-zero coefficients on the diagonal (hint similarity).

(4) Consider $V = C(\mathbb{R})$, $r > 0$ and $T : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T(f)(t) = f(t + r)$. Calculate $e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$ where convergence takes place pointwise. How would you define $T^{\frac{1}{2}}$ and e^{tT}?

Transition-hw8

Due date: November 8

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$$p_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!} A^k$$

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Transition-hw9

Due date: December 1

(1) Solve the following linear DE
   (a) $y'' = 2y' - y$, $y(0) = y'(0) = 1$;
   (b) $y'' = 2y' + y$, $y(0) = y'(0) = 1$;
   (c) $y''' = 4y'' - 5y' + y$, $y(0) = y'(0) = y''(0) = 1$

Transition-hw10

Due date: December 1

(1) Solve the following linear DE
  (a) $y'' = 2y' - y$, $y(0) = y'(0) = 1$;
  (b) $y'' = 2y' + y$, $y(0) = y'(0) = 1$;
  (c) $y''' = 4y'' - 5y' + y$, $y(0) = y'(0) = y''(0) = 1$

Transition-hw11

Due Date: Wednesday, December 8

(1) Let $F : C[0,1] \to C[0,1]$ be the function

$$F(x)(t) \;=\; \int\limits_0^t x(s)ds \;.$$

Calculate $F'(1)$ and show that $F'(1)$ has no bounded inverse. (Hint: That would imply $\sup_{0\le s\le 1} |h(s)| \;\le\; c \sup_t |\int_0^t h(s)ds|$ for com constant c. However, a clever choice such that $\int_0^t h(s)ds \;=\; \sin(nt)$ makes that pretty impossible).

(2) We want to apply the implicit function theorem for

$$F(x,y)(t) \;=\; \int_0^t x(s)y(s)ds$$

at $(x_0, y_0) \;=\; (1,1)$. Show that $D_2F(1,1)$ is not invertible and thus the implicit function theorem does not imply. Show that the solution to $F(x,y) = F(1,1)$ is given by $y(s) \;=\; \frac{1}{x(s)}$ and that this is well defined for $\|x-1\| < 1$.

(3) A better way to obtain a good solution is to consider the function

$$F(x,y)(s) \;=\; x(s)y(s)$$

Show that the implicit function theorem applies and yields a map $u : B(1,\delta) \to C[0,1]$ such that

$$F(x,u(x)) \;=\; F(1,1) \;.$$

Calculate the derivative.

(4) Show that there exists a differentiable function $f : [1-\delta, 1+\delta] \to \mathbb{R}$ satisfying

$$(1+f(x))^{\frac{1}{3}} = f(x)$$

Calculate the derivative at 1.