

1. Linear differential equations

In the vector-valued situation, Picard's iteration works equally well.

PROPOSITION 1.1. *Let I be an interval and X be a Banach space. Let $x_0 \in I$ and $y_0 \in X$ and $\delta > 0$. Let $\phi : I \times \bar{B}(y_0, \delta) \rightarrow X$ be a continuous function such that*

$$\|\phi(t, x) - \phi(t, y)\| \leq L\|x - y\| .$$

Then there exists an h such that

$$f'(t) = \phi(t, f(t)) \quad f(x_0) = y_0$$

has a unique solution among continuous functions $f : (x_0 - h, x_0 + h) \rightarrow B(y_0, \delta)$.

PROOF. Define

$$T(f)(t) = y_0 + \int_0^t \phi(s, f(s)) ds$$

Let $\alpha > L$ and $h > 0$. We consider the Banach space

$$Y = \{f : [x_0 - h, x_0 + h] \rightarrow X : f \text{ continuous} \}$$

with the norm

$$\|f\| = \sup_{|s| \leq h} e^{-\alpha|s|} \|f(x_0 + s)\| .$$

We consider the closed subset C defined by

$$C = \{f : f([x_0 - h, x_0 + h]) \subset \bar{B}(y_0, \delta)\} .$$

With the appropriate choice of h we find $T(C) \subset C$. The Banach contraction principle yields a fixpoint $T(f) = f$ which satisfies

$$f'(t) = \phi(t, f(t)) .$$

For any solution $g'(t) = \phi(t, g(t))$ with $g(x_0) = y_0$ we have

$$y_0 + \int_{x_0}^t \phi(s, g(s)) ds = y_0 + \int_{x_0}^t g'(s) ds = g(t) .$$

Thus we have uniqueness for continuous g with values in $\bar{B}(y_0, \delta)$. ■

Let us recall the usual trick for transforming DE with linear coefficients into systems: We are given the differential equation

$$\boxed{\text{1de}} \quad (1.1) \quad y^n(x) + \sum_{k=0}^{n-1} a_k y^k(x) = 0$$

with conditions $y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$. Then we introduce the new variables

$$y_0 = y, y_1 = y', \dots, y_{n-1} = y^{(n-1)}.$$

This leads to the matrix valued equation

$$\vec{y}' = A(\vec{y}), \quad \vec{y}(x_0) = (y_0, \dots, y_{n-1})$$

where

$$\boxed{\text{maa}} \quad (1.2) \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & & \\ 0 & 0 & 1 & 0 & \dots & \\ \vdots & & & & & \\ 0 & 0 & \dots & 0 & 1 & \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} & \end{pmatrix}.$$

PROPOSITION 1.2. *Let X be a Banach space and $A : X \rightarrow X$ be linear map. The differential equation*

$$f'(t) = A(t) f(t), \quad f(0) = y_0$$

has the (unique) solution

$$f(t) = e^{tA} y_0.$$

PROOF. We consider the power series

$$g(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

with values in $L(X, X)$. By Lemma ^{pow}7.10, we deduce

$$g'(t) = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k = Ag(t).$$

Therefore the chain rule implies that

$$f(t) = g(t)y_0 = e_{y_0}(g(t))$$

satisfies

$$f'(t) = e_{y_0}(g'(t)) = g'(t)y_0 = A(g(t)y_0) = A(f(t)).$$

The condition $f(0) = y_0$ is obvious. (Note that uniqueness follows from the Picard iteration method.) ■

LEMMA 1.3. Let A be a complex $n \times n$ matrix and U invertible such that $A = UBU^{-1}$. Then for every power series $f(t) = \sum_k a_k t^k$ converging on \mathbb{R} we have

$$f(A) = Uf(B)U^{-1}.$$

PROOF. Since power series are uniformly converging it suffice to show the assertion for polynomials. By linearity we only have to consider $f(t) = t^k$. However,

$$A^k = [UBU^{-1}]^k = \underbrace{UBU^{-1}UB \dots U^{-1}UBU^{-1}}_{k\text{-terms}} = UB^kU^{-1}. \quad \blacksquare$$

LEMMA 1.4. Let B an $n \times n$ -Jordan block for the eigenvalue λ . Then

$$e^{tB} = e^{t\lambda} \left(\sum_{k=0}^n \frac{t^k}{k!} \sum_{j=1, \dots, n-1-k} e_{j, j+k+1} \right)$$

where e_{ij} are the matrix units.

PROOF. We may write $B = \lambda 1 + C$ where

$$C = \sum_{j=1}^{n-1} e_{j, j+1}$$

is a sum of matrices with only one entry. Note that $e_{lr}e_{st} = \delta_{rs}e_{lt}$. Then we get

$$\begin{aligned} C^m &= \left(\sum_{j=1}^{n-1} e_{j, j+1} \right)^m \\ &= \sum_{j_1, \dots, j_m=1}^{n-1} e_{j_1, j_1+1} \cdots e_{j_m, j_m+1} \\ &= \sum_{j=1, \dots, n-1, j+m \leq n-1}^m e_{j, j+1} e_{j+1, j+2} \cdots e_{j+m, j+m+1} \\ &= \sum_{j=1, \dots, n-1-m}^m e_{j, j+m+1}. \end{aligned}$$

In particular, $C^n = 0$. This yields

$$\begin{aligned} e^{tC} &= \sum_{k=0}^{\infty} \frac{t^k C^k}{k!} \\ &= \sum_{k=0}^n \frac{t^k}{k!} \sum_{j=1, \dots, n-1-k} e_{j, j+k+1}. \end{aligned}$$

Finally we note that that for commuting matrices $e^{A+B} = e^A e^B$ follows from the properties of the binomial coefficients (as for scalars). \blacksquare

An example will illustrate the procedure

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we have

$$C^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $C^3 = 0$, we get

$$e^{tC} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the monomials $\frac{t^k}{k!}$ ‘move’ in the shifted diagonal. Note that the solutions of (8.1) should appear as linear combinations of the columns.

wrons COROLLARY 1.5. *Let A be (real) $n \times n$ matrix. Then the space of solutions*

$$y' = A(y)$$

has dimension n . The complex solutions are of the form $y(t) = e^{\lambda_j t} v(t)$ where v is a vector of polynomials of degree $< s_j$ where s_j is the power of λ_j occurring in the minimal polynomial of A . The function $g(t) = \det(e^{tA})$ is called the Wronskian and satisfies

$$g'(t) = \operatorname{tr}(A)g(t).$$

PROOF. Every solution is uniquely determined by the initial condition $y(0)$ and $y(t) = e^{At}y(0)$. By changing the initial conditions, we may assume that A is in Jordan normal form. Then the assertion on the particular form follows immediately from the considerations above. The second assertion follows from the chain rule. Indeed for $t = 0$ we have

$$g'(0) = (\det)'(1)(A) = \operatorname{tr}(A).$$

For arbitrary t we have

$$\lim_{h \rightarrow 0} \frac{\det(e^{t+h}A) - \det(e^{tA})}{h} = \lim_{h \rightarrow 0} \frac{\det(e^{hA}) - 1}{h} \det(e^{tA}) = \operatorname{tr}(A)g(t). \quad \blacksquare$$

dd LEMMA 1.6. Let A be the matrix from (8.2). Then

$$\det(A - \lambda) = (-1)^n \left(\sum_{k=0}^{n-1} a_k \lambda^k + \lambda^n \right)$$

PROOF. For $n = 2$ we have

$$\det(A - \lambda) = -\lambda(-(a_1 + \lambda)) - (-a_0) = \lambda^2 a_2 + \lambda a_1 + a_0.$$

Now, we proceed by induction and develop the determinant after the first column.

Then

$$\begin{aligned} \det(A - \lambda) &= -\lambda \det_{n-1}((A - \lambda)_{1,1}) + (-1)^{n-1}(-a_0) \\ &= (-\lambda)(-1)^{n-1} \left(\sum_{k=0}^{n-2} a_{k+1} \lambda^k + \lambda^{n-1} \right) + (-1)^n a_0 \\ &= (-1)^n \left(\sum_{k=0}^{n-1} a_k \lambda^k + \lambda^n \right). \quad \blacksquare \end{aligned}$$

PROPOSITION 1.7. Let $p(x) = \sum_{k=0}^n a_k x^k$ be a polynomial of degree n with $a_n = 1$. Let $\lambda_1, \dots, \lambda_m$ be the roots of the polynomial and

$$p(x) = \prod_{j=1}^m (x - \lambda_j)^{r_j}.$$

Then the system

$$e^{\lambda_1 t}, \dots, e^{\lambda_1 t} t^{r_1-1}, \dots, e^{\lambda_m t}, \dots, e^{\lambda_m t} t^{r_m-1}$$

is basis for the space of solutions of

$$\sum_{k=0}^n a_k y^{(k)} = 0.$$

PROOF. On the algebra $C_\infty(\mathbb{R}, \mathbb{C})$ of infinitely often differentiable maps we define the linear map

$$D(f) = f'.$$

The our problem is to find solutions for

$$p(D)f = 0.$$

Note that

$$p(x) = \prod (x - \lambda_j)^{r_j}$$

implies (after doing the math) that

$$p(D) = \prod (D - \lambda_j)^{r_j}.$$

Now, we consider $j = 1, \dots, m$ and $f_k(t) = e^{\lambda_j t} t^k$. Note that

$$\begin{aligned} (D - \lambda_j)f_k &= (D - \lambda_j)e^{\lambda_j t} t^k = (e^{\lambda_j t} t^k)' - \lambda_j e^{\lambda_j t} t^k \\ &= \lambda_j e^{\lambda_j t} t^k + e^{\lambda_j t} k t^{k-1} - \lambda_j e^{\lambda_j t} t^k = k e^{\lambda_j t} t^{k-1}. \end{aligned}$$

By induction we get

$$(D - \lambda_j)^r(f_k) = k(k-1)\cdots(k-r+1)f_{k-r}.$$

Thus for $k < r$ we always find 0. Collection all the solutions we have found n solutions y_1, \dots, y_n . Using the matrix A from [\(8.2\)](#) we may write

$$\vec{y}_j(t) = e^{tA} \vec{y}_j(0)$$

where

$$\vec{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \dots \\ y^{n-1}(t) \end{pmatrix}.$$

In order to show linear independence we recall that by Lemma [8.6](#) we know the characteristic polynomial for A and thus Corollary [8.5](#) tells is that we find a system of n linear independent solutions in the span of the solutions discovered above. Thus the solutions have to be linearly independent. ■