METHOD OF STEEPEST DESCENT - EXAMPLES

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Abstract. We will ilustrate how method of steepest descent can be used for the asymptotic evaluation of contour integrals. Three examples will be discused.

1. Introduction

The idea behing the method of steepest descent is to make the best use of the Watson's lemma:

Lemma. *Suppose* $\varphi(z)$ *is analytic in the sector in the complex plane* $0 < |t| < R$, $|argt| < \delta < \pi$ *(with possible branch point at the origin) and suppose*

$$\varphi(t) = \sum_{k=1}^{\infty} a_k t^{\frac{k}{n}-1} \qquad \textit{for} \quad |t| < R,$$

and that

$$|\varphi(t)| \leq K e^{bt} \qquad \textit{for} \quad R \leq t \leq T,$$

then

$$\int_0^T e^{-zt}\varphi(t)dt \sim \sum_{k=1}^{\infty} a_k \Gamma\big(\frac{k}{n}\big) z^{-\frac{k}{n}}$$

as $|z| \to \infty$ *in the sector* $|argz| \leq \delta < \frac{\pi}{2}$.

To take the best advantage of the above lemma it may require to deform the contour of integration. Then we would like to apply Laplace's method and use the fact that for large x values of e^{-x} are very small. The method allows to find asymtotic behavior for large x for the integral

$$f(x) = \int_a^b e^{xh(t)}\varphi(t)dt. \qquad (\star)$$

However, the problem appears: analytic functions on $\mathbb{C}$ may only have saddle points, they do not have local minimas or local maximas. Since we have:

Theorem. *Maximum Modulus Principle If* $f(z)$ *is analytic on the interior of a region, then* $|f(z)|$ *cannot attain strict local maximum on the interior of the region.*

Proof. Suppose z is the location of local maximum of $|f(z)|$, so for all w in a neighborhood of z we have $|f(z)| \geq |f(w)|$. If B is a disk around z lying in the neighborhood and having radius R, then by the Mean Value Theorem,

$$|f(z)| \leq \frac{1}{\pi R^2}\int_B |f(w)|dw.$$

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Since $|f(z)| \geq |f(w)|$, the above can hold if and only if $|f(z)| = |f(w)|$ for all w in the disk. Then, since $|f(z)|$ is constant everywhere in the disk B, Cauchy - Riemann conditions imply that $f(z)$ is constant. Thus, if $|f(z)|$ attains its maximum on the interior of some region, it does so only because $f(z)$ is a constant function. □

We are looking for a path along which Laplace's method can be applied. We deform the path of integration in such a way, that the real part of $h(t)$ in $(\star)$ decreases as rapidly as possible on the new path. The new path is called the path of steepest descent, for real valued functions is along gradient directions. For a complex valued function $f(z) = u(x, y) + iv(x, y)$ we know that the Cauchy-Riemann conditions force $\bigtriangledown u \cdot \bigtriangledown v = 0$. So curves $Imf = const$ are the steepest paths for Ref, and curves $Ref = const$ are the steepest paths for Imf.

2. Examples

Example 1. Hankel's Integral

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C e^s s^{-z} ds$$

where the contour C traverses from $s = -i\epsilon - \infty$ to $s = i\epsilon - \infty$ around branch cut on $Res < 0$ axis, $\epsilon > 0$.
$e^s = e^{Res+Ims} = e^{Res}(cos(Ims) + isin(Ims))$, defined uniquely,
z - fixed, then $ln(s^{-z}) = -zlns = -z(ln|s| + iarg(s))$ so $s^{-z} = e^{ln(s^{-z})} =$
$= e^{-z(ln|s|+iarg(s))} = |s|^{-z}e^{-izarg(s)}$
$arg(s)$ is not unique, branch point is $s = 0$. Need a branch cut, let $\theta \in (-n\pi, n\pi)$, so set the branch cut at $\theta = \pi$.

Consider change of variables $s = zt$ to make it possible to apply Laplace's method. Then:
$s = zt \Rightarrow t = \frac{s}{z}$ and this change won't affect the contour C, $ds = zdt$. So

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C e^{zt}(zt)^{-z} z dt =$$
$$= \frac{1}{2\pi i} z^{-z+1} \int_C e^{zt} t^{-z} dt = \frac{1}{2\pi i} z^{1-z} \int_C e^{zt-zlnt} dt = \frac{1}{2\pi i} z^{1-z} \int_C e^{z(t-lnt)} dt$$

$f(t) = t - lnt$, f is analytic away from $Ret \leq 0$
$\frac{\partial f}{\partial t} = 1 - \frac{1}{t} = \frac{t-1}{t}$, $\frac{\partial f}{\partial t} = 0$ for $t = 1$.
Since f is analytic at $t = 1$, it must be a saddle point.
$f(1) = 1 - ln1 = 1 - (ln|1| + i0) = 1$

$f(t) = t - lnt$ $\quad f(1) = 1$
$f^{(1)}(t) = 1 - \frac{1}{t}$ $\quad f^{(1)}(1) = 0$
$f^{(2)}(t) = \frac{1}{t^2}$ $\quad f^{(2)}(1) = 1$
$f^{(3)}(t) = -\frac{2}{t^3}$ $\quad f^{(3)}(1) = -2$
$f^{(4)}(t) = \frac{6}{t^4}$ $\quad f^{(4)}(1) = 6$
... ...
$f^{(n)}(t) = \frac{(-1)^n(n-1)!}{t^n}$ $\quad f^{(n)}(1) = (-1)^n(n-1)!$

So near $t = 1$

$$f(t) = 1 + 0 + \sum_{n=2}^{\infty} \frac{1}{n!}(-1)^n(n-1)!(t-1)^n = 1 + \sum_{n=2}^{\infty} \frac{1}{n}(1-t)^n$$

At the point $t = 1$, $Ref(1) = 1$, $Imf(1) = 0$, so the steepest descent path through the saddle point must satisfy $Imf(t) = const = 0$, i.e. $Im(t - lnt) = 0$.
Set $t = x+iy$. Then $t-lnt = x+iy-ln(x+iy) = x+iy-(ln|x^2+y^2|+iarctan\frac{y}{x})$.
So $Im(t - lnt) = y - arctan\frac{y}{x} = 0$.
$y = arctan\frac{y}{x} \Rightarrow tany = \frac{y}{x} \Rightarrow x = ycoty.$

Since near $t = 1$, $t - lnt$ is localy quadratic, we are looking for such a change of variables, that curve, parametrized by η, passing through $t = 0$ at $\eta = 0$, with $Imf = 0$, $Ref < 1$, has quadratic behavior in η.
Consider $f(t) = 1 - \eta^2$, $\eta \in \mathbb{R}$, then

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} z^{1-z} \int_{-\infty}^{\infty} e^{z(1-\eta^2)} (\frac{dt}{d\eta}) d\eta = \frac{1}{2\pi i} z^{1-z} e^z \int_{-\infty}^{\infty} e^{z\eta^2} (\frac{dt}{d\eta}) d\eta$$

We need to find an approximation for $\frac{dt}{d\eta}$. Since $t - lnt = 1 - \eta^2$ we get:
$t(\eta) = 1 + \sqrt{2}\eta i - \frac{2}{3}\eta^2 - \frac{i}{9\sqrt{2}}\eta^3 - \frac{2}{135}\eta^4 + \frac{i}{540\sqrt{2}}\eta^5 - \frac{4}{8505}\eta^6 + ...$
by computer algebra system - Scientific Workspace®. Hence:
$\frac{dt}{d\eta} = \sqrt{2}i - \frac{4}{3}\eta - \frac{i}{3\sqrt{2}}\eta^2 - \frac{8}{135}\eta^3 + \frac{i}{108\sqrt{2}}\eta^4 - \frac{24}{8505}\eta^5 + ...$

We can integrate term by term applying Watson's Lemma, since the series is convergent (has finite radius of convergence). So:

$$\frac{1}{\Gamma(z)} = \frac{1}{\sqrt{2}\pi i} z^{1-z} e^z \int_{-\infty}^{\infty} e^{z\eta^2} \left(i - \frac{4}{3\sqrt{2}}\eta - \frac{i}{6}\eta^2 - \frac{8}{135\sqrt{2}}\eta^3 + \frac{i}{216}\eta^4 - \frac{24}{8505\sqrt{2}}\eta^5 + ... \right) d\eta =$$

$$= \frac{1}{\sqrt{2}\pi i} z^{1-z} e^z \left[i \int_{-\infty}^{\infty} e^{z\eta^2} d\eta - \frac{4\sqrt{2}}{6} \int_{-\infty}^{\infty} e^{z\eta^2} \eta d\eta - \frac{i}{6} \int_{-\infty}^{\infty} e^{z\eta^2} \eta^2 d\eta + \frac{8\sqrt{2}}{270} \int_{-\infty}^{\infty} e^{z\eta^2} \eta^3 d\eta + \right.$$

$$\left. + \frac{i}{216} \int_{-\infty}^{\infty} e^{z\eta^2} \eta^4 d\eta - \frac{12\sqrt{2}}{8505} \int_{-\infty}^{\infty} e^{z\eta^2} \eta^5 d\eta + ... \right]$$

After eliminating all the integrals of odd functions (they are 0, since the region of integrating is symmetrical around 0), we get:

$$\frac{1}{\Gamma(z)} = \frac{1}{\sqrt{2}\pi i} z^{1-z} e^z \left[i \int_{-\infty}^{\infty} e^{z\eta^2} d\eta - \frac{i}{6} \int_{-\infty}^{\infty} e^{z\eta^2} \eta^2 d\eta + \frac{i}{216} \int_{-\infty}^{\infty} e^{z\eta^2} \eta^4 d\eta + ... \right]$$

Now all integrals contain i , so we may cancel with i in the front:

$$\frac{1}{\Gamma(z)} = \frac{1}{\sqrt{2}\pi} z^{1-z} e^z \left[\int_{-\infty}^{\infty} e^{z\eta^2} d\eta - \frac{1}{6} \int_{-\infty}^{\infty} e^{z\eta^2} \eta^2 d\eta + \frac{1}{216} \int_{-\infty}^{\infty} e^{z\eta^2} \eta^4 d\eta + ... \right]$$

So the first approximation is:

$$\frac{1}{\Gamma(z)} \approx \frac{1}{\sqrt{2}\pi} z^{1-z-0.5} e^z \sqrt{\pi} = \frac{1}{\sqrt{2\pi}} e^z z^{0.5-z},$$

which is know for $\Gamma(z) \approx \sqrt{2\pi} e^{-z} z^{z-0.5}$ as Stirling's Formula. Next approximation we get after evaluating the second integral:

$$\int_{-\infty}^{\infty} e^{z\eta^2} \eta^2 d\eta = \left[-\frac{\eta}{2z} e^{-z\eta^2} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{2z} e^{-z\eta^2} d\sqrt{z}\eta = 0 + \frac{1}{2z} \frac{1}{\sqrt{z}} \sqrt{\pi}$$

So

$$\frac{1}{\Gamma(z)} \approx \frac{1}{\sqrt{2}\pi} z^{1-z} e^z \left(\sqrt{\frac{\pi}{z}} - \frac{1}{12z} \sqrt{\frac{\pi}{z}} \right) = \frac{1}{2\sqrt{\pi}} z^{1-z} e^z \left(\frac{12z-1}{12z} \right) = \frac{1}{24\sqrt{\pi}} z^{-z} e^z (12z-1)$$

For large z we have: $\Gamma(z) \approx 24\sqrt{\pi} z^z e^{-z} \frac{1}{12z-1}$.

Example 2.

$$F(x) = \int_0^1 exp[ix(t+t^2)] \frac{dt}{\sqrt{t}} \qquad \text{for large } x$$

Path of integration is given at the picture:

First look at the saddle point of $f(t) = i(t+t^2)$

$\frac{\partial f}{\partial t} = i(1+2t)$, $\frac{\partial f}{\partial \bar{t}} = 0$ so f is analytic.
$\frac{\partial f}{\partial t} = 0$ iff $t = -\frac{1}{2}$, $f(-\frac{1}{2}) = -\frac{i}{4}$
$Ref(-\frac{1}{2}) = 0$, $Imf(-\frac{1}{2}) = -\frac{1}{4}$, so either $Ref(t) = 0$, or $Imf(t) = -\frac{1}{4}$.
Let $t = x + iy$:
$f(x+iy) = i(x+iy+x^2+2ixy-y^2) = -y-2xy+i(x+x^2-y^2)$
$Re(t) = 0 \iff y+2xy = 0$ so $y = 0$ or $x = -\frac{1}{2}$
$Imf(t) = -\frac{1}{4} \iff x+x^2-y^2 = -\frac{1}{4}$

$$(x+\frac{1}{2})^2 - y^2 = 0$$

$$(x+y+\frac{1}{2})(x-y+\frac{1}{2}) = 0$$

So $y = -x-\frac{1}{2}$ or $y = x+\frac{1}{2}$
Hence Ref decreaces/increases rapidly along $y = -x-\frac{1}{2}$ and $y = x+\frac{1}{2}$

There is a problem with choosing a branch cut for $\sqrt{z}$. If we put branch cut somewhere starting from the origin and going to $-\infty$ on Ret, then branch cut will cut the lines $Imf(t) = -\frac{1}{4}$. If we take branch starting at the origin and having positive real part, we can't close the contour at ∞. So the "saddle point" method is not working here and we have to try another approach.
Let's look for the steepest descent paths going through $t = 0$ and $t = 1$, the ends of the contour.
At $t = 0$ we have $f(t) = 0$, so $Imf(t) = 0 \Rightarrow x+x^2-y^2 = 0$
So the first path is $y^2 = x^2 + x$. Let's call it C_1.
At $t = 1$ we have $f(t) = i(1+1) = 2i$, so $Imf(t) = 2 \Rightarrow x+x^2-y^2 = 2$
So the second path is $x+x^2-y^2 = 2$. Let's call it C_2.
Hence we deform the original path of integration into two, C_1 and C_2. We introduce convienient change of variables, so $f(t)$ is:
on C_1: $i(t+t^2) = f(0)-\tau = 0-\tau = -\tau$ and $0 \le \tau < +\infty$ "decreasing" path
on C_2: $i(t+t^2) = f(1)-\eta = 2i-\eta$ and $0 \le \eta < +\infty$ "increasing" path

on C_1	on C_2
$t+t^2=i\tau$	$t+t^2=2+i\eta$
$(t+\frac{1}{2})^2=i\tau+\frac{1}{4}$	$(t+\frac{1}{2})^2=2+\frac{1}{4}+i\eta$
$t+\frac{1}{2}=\sqrt{i\tau+\frac{1}{4}}$	$t+\frac{1}{2}=\sqrt{\frac{9}{4}+i\eta}$
$t=-\frac{1}{2}+\sqrt{i\tau+\frac{1}{4}}$	$t=-\frac{1}{2}+\sqrt{\frac{9}{4}+i\eta}$
$t=-\frac{1}{2}+\frac{1}{2}\sqrt{4i\tau+1}$	$t=-\frac{1}{2}+\frac{3}{2}\sqrt{\frac{4}{9}i\eta+1}$
$dt=\frac{1}{2}\frac{1}{2\sqrt{4i\tau+1}}(4i)d\tau$	$dt=\frac{3}{2}\frac{1}{2\sqrt{\frac{4}{9}i\eta+1}}\frac{4}{9}id\eta$
$dt=\frac{i}{\sqrt{1+4i\tau}}d\tau$	$dt=\frac{i}{3\sqrt{\frac{4}{9}i\eta+1}}d\eta$

$$F(x)=\int_{C_1}\frac{e^{-x\tau}}{\left(-\frac{1}{2}+\frac{1}{2}\sqrt{4i\tau+1}\right)^{\frac{1}{2}}}\frac{id\tau}{\sqrt{4i\tau+1}}+\int_{C_2}\frac{e^{x(2i-\eta)}}{\left(-\frac{1}{2}+\frac{3}{2}\sqrt{\frac{4}{9}i\eta+1}\right)^{\frac{1}{2}}}\frac{id\eta}{3\sqrt{\frac{4}{9}i\eta+1}}=$$

$$=i\int_0^\infty\frac{e^{-x\tau}d\tau}{\left(-\frac{1}{2}+\frac{1}{2}\sqrt{4i\tau+1}\right)^{\frac{1}{2}}\sqrt{4i\tau+1}}+\frac{i}{3}e^{2ix}\int_0^\infty\frac{e^{-x\eta}d\eta}{\left(-\frac{1}{2}+\frac{3}{2}\sqrt{\frac{4}{9}i\eta+1}\right)^{\frac{1}{2}}\sqrt{\frac{4}{9}i\eta+1}}$$

Now in order to aplly Watson's lemma, we need to expand the functions

$$g_1(\tau)=\left(-\frac{1}{2}+\frac{1}{2}\sqrt{4i\tau+1}\right)^{-\frac{1}{2}}(4i\tau+1)^{-\frac{1}{2}}$$

and

$$g_2(\eta)=\left(-\frac{1}{2}+\frac{3}{2}\sqrt{\frac{4}{9}i\eta+1}\right)^{-\frac{1}{2}}\left(\frac{4}{9}i\eta+1\right)^{-\frac{1}{2}}$$

into power series and integrate term by term. To do so we use computer algebra system - Scientific Workspace® and arrive at:

$$
\begin{aligned}
g_1(\tau) = {} & \frac{1}{\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)\sqrt{\tau}} - \frac{3}{2}\frac{i}{\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\sqrt{\tau} - \frac{35}{8\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)}\tau^{\frac{3}{2}} + \frac{231}{16}\frac{i}{\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\tau^{\frac{5}{2}} + \\
& + \frac{6435}{128\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)}\tau^{\frac{7}{2}} + -\frac{46\,189}{256}\frac{i}{\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\tau^{\frac{9}{2}} - \frac{676\,039}{1024\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)}\tau^{\frac{11}{2}} + \\
& + \frac{5014\,575}{2048}\frac{i}{\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\tau^{\frac{13}{2}} + \frac{300\,540\,195}{32\,768\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)}\tau^{\frac{15}{2}} - \frac{2268\,783\,825}{65\,536}\frac{i}{\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\tau^{\frac{17}{2}} + \\
& - \frac{34\,461\,632\,205}{262\,144\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)}\tau^{\frac{19}{2}} + \frac{263\,012\,370\,465}{524\,288}\frac{i}{\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\tau^{\frac{21}{2}} + \frac{8061\,900\,920\,775}{4194\,304\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)}\tau^{\frac{23}{2}} + \\
& - \frac{61\,989\,816\,618\,513}{8388\,608}\frac{i}{\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\tau^{\frac{25}{2}} - \frac{956\,086\,325\,095\,055}{33\,554\,432\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)}\tau^{\frac{27}{2}} + \\
& + \frac{7391\,536\,347\,803\,839}{67\,108\,864}\frac{i}{\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\tau^{\frac{29}{2}} + \frac{916\,312\,070\,471\,295\,267}{2147\,483\,648\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)}\tau^{\frac{31}{2}} + \\
& - \frac{7113\,260\,368\,810\,144\,185}{4294\,967\,296}\frac{i}{\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\tau^{\frac{33}{2}} - \frac{110\,628\,135\,069\,209\,194\,801}{17\,179\,869\,184\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)}\tau^{\frac{35}{2}} + O\left(\tau^{\frac{37}{2}}\right) \\
g_2(\eta) = {} & 1 + \left(-\frac{7}{18}i\right)\eta - \frac{37}{216}\eta^2 + \frac{911}{11\,664}i\eta^3 + \frac{30\,395}{839\,808}\eta^4 + \left(-\frac{28\,441}{1679\,616}i\right)\eta^5 + \\
& - \frac{1446\,473}{181\,398\,528}\eta^6 + \frac{12\,320\,597}{3265\,173\,504}i\eta^7 + \frac{280\,927\,867}{156\,728\,328\,192}\eta^8 + \left(-\frac{21\,688\,120\,685}{25\,389\,989\,167\,104}i\right)\eta^9 + \\
& - \frac{373\,104\,869\,137}{914\,039\,610\,015\,744}\eta^{10} + \frac{1072\,358\,929\,187}{5484\,237\,660\,094\,464}i\eta^{11} + \frac{111\,197\,363\,140\,591}{1184\,595\,334\,580\,404\,224}\eta^{12} + \\
& - \frac{962\,772\,915\,749\,701}{21\,322\,716\,022\,447\,276\,032}i\eta^{13} - \frac{618\,593\,352\,697\,865}{28\,430\,288\,029\,929\,701\,376}\eta^{14} + \frac{16\,123\,828\,045\,741\,351}{1535\,235\,553\,616\,203\,874\,304}i\eta^{15} + \\
& + \frac{2244\,942\,961\,329\,191\,467}{442\,147\,839\,441\,466\,715\,799\,552}\eta^{16} + \left(-\frac{6521\,267\,830\,194\,891\,391}{2652\,887\,036\,648\,800\,294\,797\,312}i\right)\eta^{17} + \\
& - \frac{1024\,338\,534\,463\,140\,293\,063}{859\,535\,399\,874\,211\,295\,514\,329\,088}\eta^{18} + \frac{8950\,298\,886\,900\,372\,253\,175}{15\,471\,637\,197\,735\,803\,319\,257\,923\,584}i\eta^{19} + O\left(\eta^{20}\right)
\end{aligned}
$$

To approximate F we use just a few terms from the power series expantions of g_1 and g_2:
$g_1(\tau) = \frac{\sqrt{2}}{1+i}\frac{1}{\sqrt{\tau}}\left(1 - \frac{3}{2}i\tau - \frac{35}{8}\tau^2 + \frac{231}{16}i\tau^3 + ...\right)$ and

$g_2(\eta) = 1 - \frac{7}{18}i\eta - \frac{37}{216}\eta^2 + \frac{911}{11664}i\eta^3 + ...$

$$F(x) = i\int_o^\infty e^{-x\tau}\frac{\sqrt{2}}{1+i}\frac{1}{\sqrt{\tau}}\left(1 - \frac{3}{2}i\tau - \frac{35}{8}\tau^2 + \frac{231}{16}i\tau^3 + ...\right)d\tau + \\ + \frac{i}{3}e^{2ix}\int_0^\infty e^{-x\eta}\left(1 - \frac{7}{18}i\eta - \frac{37}{216}\eta^2 + \frac{911}{11664}i\eta^3 + ...\right)d\eta$$

Observe that $\frac{i}{1+i}\sqrt{2} = \frac{1+i}{\sqrt{2}}$ and $\left(\frac{1+i}{\sqrt{2}}\right)^2 = \frac{1+2i-1}{2} = i$, hence $\frac{1+i}{\sqrt{2}} = \sqrt{i}$. So:

$$F(x) = \sqrt{i}\int_0^\infty e^{-x\tau}\frac{1}{\sqrt{\tau}}\left(1 - \frac{3}{2}i\tau - \frac{35}{8}\tau^2 + \frac{231}{16}i\tau^3 + ...\right)d\tau + \\ + \frac{i}{3}e^{2ix}\int_0^\infty e^{-x\eta}\left(1 - \frac{7}{18}i\eta - \frac{37}{216}\eta^2 + \frac{911}{11664}i\eta^3 + ...\right)d\eta$$

Now we integrate term by term to obtain:

$$F(x) = \sqrt{\frac{\pi}{x}}e^{i\frac{\pi}{4}}\left(1 - \frac{3i}{4x} - \frac{108}{32x^2} + \frac{3465i}{128x^3} + ...\right) - \frac{i}{3}e^{2ix}\left(\frac{1}{x} - \frac{7i}{18x^2} - \frac{37}{108x^3} + ...\right)$$

Hence

$$\int_0^1 exp[ix(t+t^2)]\frac{dt}{\sqrt{t}} \approx \sqrt{\pi}e^{i\frac{\pi}{4}}x^{-0.5} - \frac{i}{3}e^{2ix}x^{-1} - \sqrt{\pi}\frac{3i}{4}e^{i\frac{\pi}{4}}x^{-1.5} + \frac{i}{3}e^{2ix}\frac{7i}{18}x^{-2} - \sqrt{\pi}e^{i\frac{\pi}{4}}\frac{108}{32}x^{-2.5} + \\ + \frac{i}{3}e^{2ix}\frac{37}{108}x^{-3} + \sqrt{\pi}e^{i\frac{\pi}{4}}\frac{3465i}{128}x^{-3.5} + O\left(x^{-4}\right)$$

for large x.

Example 3.

$$F(z) = \int_0^1 e^{-zt^3}dt$$

First, if $|z|$ is small, we can expand e^{-zt^3} in a Taylor series around 0:

$$e^{-zt^3} = \sum_{n=0}^{\infty}\frac{(-zt^3)^n}{n!}$$

and then integrate term by term:

$$\int_0^1 \frac{(-zt^3)^n}{n!}dt = \frac{(-z)^n}{n!}\int_0^1 t^{3n}dt = \frac{(-z)^n}{n!}\frac{1}{3n+1}\left[t^{3n+1}\right]_0^1 = \frac{(-z)^n}{n!}\frac{1}{3n+1}$$

Then $F(z) = \int_0^1 \sum_{n=0}^{\infty} \frac{(-zt^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\frac{1}{3n+1}$

For any fixed z the series $\sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\frac{1}{3n+1}$ converges, since $\frac{z^n}{n!}\frac{1}{3n+1} \xrightarrow[n\to\infty]{} 0$ and the series is alternating.
The problem is that for large $|z|$ the series $\sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\frac{1}{3n+1}$ converges slowly, so it is a good approximation only for z with small $|z|$. We can observe the difference between $z = 0.05$ and $z = 5$ by comparing the tables from Excel ® attached after last page.

For large $|z|$ we will use the steepest descent method. Let $z = |z|e^{i\theta}$, where $0 \le \theta < 2\pi$. Then:

$$F(z) = \int_0^1 e^{-|z|e^{i\theta}t^3} dt$$

Set $h(t) = -e^{i\theta}t^3$ and let us find the saddle points of h.
$h'(t) = -e^{i\theta}3t^2$, $h'(t) = 0$ for $t = 0$. h is analytic, so $t = 0$ is a saddle point for h.
Write $t = re^{i\varphi}$, then $h(t) = -r^3e^{i(\theta+3\varphi)}$.
$h(0) = 0 \Longrightarrow Im(h(t)) = 0$

$$Im(-r^3e^{i(\theta+3\varphi)}) = 0$$
$$Im(-r^3(cos(\theta+3\varphi) + isin(\theta+3\varphi))) = 0$$
$$sin(\theta+3\varphi) = 0$$
$$\theta + 3\varphi = k\pi, \qquad k \in \mathbb{Z}$$
$$\varphi = \frac{k\pi - \theta}{3}$$

$\varphi \in [0, 2\pi) \Longrightarrow k = 0, 1, 2, 3, 4, 5.$
Now

$$Re(h(t)) = Re(-r^3(cos(\theta+3\varphi) + isin(\theta+3\varphi))) = -r^3cos(\theta+3\varphi) =$$
$$= -r^3cos(\theta + k\pi - \theta) = -r^3cos(k\pi)$$

For $k = 0, 2, 4$, $cos(k\pi) = 1 \Longrightarrow Re(h(t)) \ge 0$
For $k = 1, 3, 5$, $cos(k\pi) = -1 \Longrightarrow Re(h(t)) \le 0$

Therefore the radial lines $t = rexp\left(i\frac{k\pi-\theta}{3}\right)$, $r \ge 0$, are for $k = 0, 2, 4$ the steepest descent paths, and for $k = 1, 3, 5$ they are the steepest ascent paths.
Now we need to find the steepest descent path through the point $t = 1$. Observe that $h(1) = -e^{i\theta}$, so $Im(h(1)) = -sin\theta$. Hence $Im(h(t)) = -r^3sin(\theta+3\varphi) = -sin\theta$,

$$r^3sin(\theta+3\varphi) = sin\theta$$

The exact path can be found after specifying θ.
Here we concider the cases when $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$.

Case $\theta = \frac{\pi}{6}$
The radial lines through saddle point $t = 0$ are: $t_1 = rexp\left(-i\frac{\pi}{18}\right)$, $t_2 = rexp\left(i\frac{11\pi}{18}\right)$, and $t_3 = rexp\left(i\frac{23\pi}{18}\right)$.
The steepest descent path through $t = 1$: $r^3 sin\left(\frac{\pi}{6} + 3\varphi\right) = sin\frac{\pi}{6} = \frac{1}{2}$
$r = (cos3\varphi + \sqrt{3}sin3\varphi)^{-\frac{1}{3}}$

Case $\theta = \frac{\pi}{4}$
The radial lines through saddle point $t = 0$ are: $t_1 = rexp\left(-i\frac{\pi}{12}\right)$, $t_2 = rexp\left(i\frac{7\pi}{12}\right)$, and $t_3 = rexp\left(i\frac{5\pi}{4}\right)$.
The steepest descent path through $t = 1$: $r^3 sin\left(\frac{\pi}{4} + 3\varphi\right) = sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$r = (cos3\varphi + sin3\varphi)^{-\frac{1}{3}}$

Case $\theta = \frac{\pi}{3}$
The radial lines through saddle point $t = 0$ are: $t_1 = rexp\left(-i\frac{\pi}{9}\right)$, $t_2 = rexp\left(i\frac{5\pi}{9}\right)$, and $t_3 = rexp\left(i\frac{11\pi}{9}\right)$.
The steepest descent path through $t = 1$: $r^3 sin\left(\frac{\pi}{3} + 3\varphi\right) = sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$r = \left(\frac{\sqrt{3}}{\sqrt{3}cos3\varphi + sin3\varphi}\right)^{\frac{1}{3}}$

In order to deform the path of integration into the steepest descent path we change variables as follows:
on C_1: $e^{i\theta}t^3 = \tau$, $0 \le \tau < \infty$
on C_2: $e^{i\theta}t^3 = e^{i\theta} + \eta$, $0 \le \eta < \infty$
So:

$$F(z) = \int_0^\infty e^{-|z|\tau}\left(\frac{dt}{d\tau}\right)d\tau + \int_0^\infty e^{-|z|(e^{i\theta}+\eta)}\left(\frac{dt}{d\eta}\right)d\eta$$

We need to find $\frac{dt}{d\tau}$ and $\frac{dt}{d\eta}$:

on C_1	on C_2
$e^{i\theta}t^3 = \tau$	$e^{i\theta}t^3 = e^{i\theta} + \eta$
$t^3 = \tau e^{i\theta}$	$t^3 = 1 + \eta e^{-i\theta}$
$t = (\tau e^{i\theta})^{\frac{1}{3}}$	$t = (1 + \eta e^{-i\theta})^{\frac{1}{3}}$
$dt = \frac{1}{3}(\tau e^{i\theta})^{-\frac{2}{3}}e^{-i\theta}d\tau$	$dt = \frac{1}{3}(1 + \eta e^{-i\theta})^{-\frac{2}{3}}e^{-i\theta}d\eta$

Therefore:

$$F(z) = \frac{e^{-i\theta}}{3}\int_0^\infty e^{-|z|\tau}(\tau e^{-i\theta})^{-\frac{2}{3}}d\tau + \frac{e^{-i\theta}}{3}\int_0^\infty e^{-|z|(e^{i\theta}+\eta)}(1+\eta e^{-i\theta})^{-\frac{2}{3}}d\eta$$

$$F(z) = \frac{(e^{-i\theta})^{\frac{1}{3}}}{3}\int_0^\infty e^{-|z|\tau}\tau^{-\frac{2}{3}}d\tau + \frac{e^{-i\theta}}{3}\int_0^\infty e^{-|z|(e^{i\theta}+\eta)}(1+\eta e^{-i\theta})^{-\frac{2}{3}}d\eta$$

For the first integral:

$$\int_0^\infty e^{-|z|\tau}\tau^{-\frac{2}{3}}d\tau = \int_0^\infty e^{-|z|\tau}\,(|z|\tau)^{\frac{1}{3}-1}\,|z|^{\frac{2}{3}}|z|^{-1}d(|z|\tau) = |z|^{-\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)$$

For the second integral:
We will use the binomial expansion:

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+1)x^n}{\Gamma(\alpha+1-n)n!}$$

which is valid for all x if α is integer, for $|x| > 1$ if α is not an integer.
We have $\alpha = -\frac{2}{3}$ and $x = e^{-i\theta}\eta$, $|x| = |\eta|$. Therefore:

$$\int_0^\infty e^{-|z|(e^{i\theta}+\eta)}(1+\eta e^{-i\theta})^{-\frac{2}{3}}d\eta \approx \int_0^\infty e^{-|z|(e^{i\theta}+\eta)} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{3})(e^{i\theta}\eta)^n}{\Gamma(\frac{1}{3}-n)n!}d\eta =$$

$$= \Gamma\left(\frac{1}{3}\right) \sum_{n=0}^{\infty} e^{-in\theta} \int_0^\infty e^{-|z|(e^{i\theta}+\eta)}\eta^n d\eta \frac{1}{n!\Gamma(\frac{1}{3}-n)} =$$

$$= \Gamma\left(\frac{1}{3}\right) e^{-|z|e^{i\theta}} \sum_{n=0}^{\infty} e^{-in\theta}\frac{1}{n!}\frac{1}{\Gamma\left(\frac{1}{3}-n\right)|z|^n}\int_0^\infty e^{-|z|\eta}(|z|\eta)^{n+1-1}\frac{d|z|\eta}{|z|} =$$

$$= \Gamma\left(\frac{1}{3}\right) e^{-|z|e^{i\theta}} \sum_{n=0}^{\infty} e^{-in\theta}\frac{1}{n!}\frac{1}{\Gamma\left(\frac{1}{3}-n\right)|z|^{n+1}}\Gamma(n+1) =$$

$$= \Gamma\left(\frac{1}{3}\right) e^{-|z|e^{i\theta}} \sum_{n=0}^{\infty} e^{-in\theta}\frac{1}{\Gamma\left(\frac{1}{3}-n\right)|z|^{n+1}}$$

and the last equality holds since $\Gamma(n+1) = n!$.
Hence with $z = |z|e^{i\theta}$ and $\Gamma(a) = (a-1)\Gamma(a-1)$ we get:

$$F(z) = |z|^{-\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)\frac{(e^{-i\theta})^{\frac{1}{3}}}{3} + \frac{e^{-i\theta}}{3}\Gamma\left(\frac{1}{3}\right)e^{-|z|e^{i\theta}}\sum_{n=0}^{\infty} e^{-in\theta}\frac{1}{\Gamma\left(\frac{1}{3}-n\right)|z|^{n+1}} =$$

$$= |z|^{-\frac{1}{3}}e^{-\frac{i\theta}{3}}\Gamma\left(\frac{4}{3}-1\right)\left(\frac{4}{3}-1\right) + \Gamma\left(\frac{4}{3}-1\right)\left(\frac{4}{3}-1\right)e^{-|z|e^{i\theta}}\sum_{n=0}^{\infty}\frac{e^{-i(n+1)\theta}}{|z|^{n+1}\Gamma\left(\frac{1}{3}-n\right)} =$$

$$= |z|^{-\frac{1}{3}}e^{-\frac{i\theta}{3}}\Gamma\left(\frac{4}{3}\right) + \Gamma\left(\frac{4}{3}\right)e^{-z}\sum_{n=0}^{\infty}\frac{|z|^{-(n+1)}e^{-i(n+1)\theta}}{\Gamma\left(\frac{1}{3}-n\right)} =$$

$$= |z|^{-\frac{1}{3}}\left(\frac{|z|}{z}\right)^{\frac{1}{3}}\Gamma\left(\frac{4}{3}\right) + \Gamma\left(\frac{4}{3}\right)e^{-z}\sum_{n=0}^{\infty}\frac{z^{-(n+1)}}{\Gamma\left(\frac{1}{3}-n\right)} =$$

$$= \frac{1}{z^{\frac{1}{3}}}\Gamma\left(\frac{4}{3}\right) + \Gamma\left(\frac{4}{3}\right)e^{-z}\sum_{n=0}^{\infty}\frac{z^{-(n+1)}}{\Gamma\left(\frac{1}{3}-n\right)}$$

So

$$F(z) \approx z^{-\frac{1}{3}}\Gamma\left(\frac{4}{3}\right) + \Gamma\left(\frac{4}{3}\right)e^{-z}\sum_{n=0}^{\infty}\frac{z^{-(n+1)}}{\Gamma\left(\frac{1}{3}-n\right)}$$

and this approximation is valid for all z with $|z|$ large.

Attachments: There are five pages of attachments.
A. one page with tables from Excel®,
B. pages 1. 2. 3. 4. with pictures to the Example #3.

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