MATH468 - BANACH SPACES - FALL 03
LEWIS' THEOREM

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1. Some Facts about Ellipsoids

Here E is an n-dimensional normed space.

Definition 1. An ellipsoid in E is a subset D of E, which is the image of the canonical Euclidean ball $B_{l_2^n}$ by a linear isomorphism.
$D = u\left(B_{l_2^n}\right)$ for some invertible $u : \mathbb{R}^n \to E$

John's question:
Is there an ellipsoid of maximal volume included in the unit ball B_E? And if so, is it unique?
By compactness of $B_{l_2^n}$ such an ellipsoid exists (trivially). John proved that such an ellipsoid is unique, denote it by D_E^{max}. Also,

$$D_E^{max} \subset B_E \subset \sqrt{n} D_E^{max}$$

which implies $d(E, l_2^n) \leq \sqrt{n}$.

$$d(E, l_2^n) = \inf\{\ \|T\|\ \|T^{-1}\| \quad |\ T : E \to l_2^n \text{ isomorphism}\}$$

By duality, there is a unique ellipsoid of minimal volume containing B_E, denote by D_E^{min}, and

$$\frac{1}{\sqrt{n}} D_E^{min} \subset B_E \subset D_E^{min}$$

The ellipsoid of maximal volume $D_E^{max} \subset B_E$ corresponds to an operator $u : l_2^n \to E$ with $\|u\| \leq 1$ such that $vol\left(u\left(B_{l_2^n}\right)\right)$ is maximal.
Since E is finite dimensional, there is only one notion of volume on E up to a multiplicative constant. If E is equipped with a fixed linear basis, we may consider the determinant $det(u)$ of any linear map $u : \mathbb{R}^n \to E$. Then for some constant $c > 0$:

$$vol\left(u\left(B_{l_2^n}\right)\right) = c|det(u)|$$

Idea: consider the operator u which maximizes $|det(u)|$, as u runs over all operators such that $|||u||| \leq 1$, where $|||\cdot|||$ is an arbitrary norm on $B\left(l_2^n, E\right)$.

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2. Trace Duality

Definition 2. The trace duality is given by the following

$$(\forall u \in \mathcal{L}(\mathbb{R}^n, E))\,(\forall v \in \mathcal{L}(E, \mathbb{R}^n)) < v, u >= tr(vu)$$

Now let $e_1, \ldots, e_n$ be a canonical basis of $\mathbb{R}^n$.
To any $u \in \mathcal{L}(\mathbb{R}^n, E)$ we can associate $x_1, \ldots, x_n$ in E by $x_i = u(e_i)$. Similarly, to any v in $\mathcal{L}(E, \mathbb{R}^n)$ we can associate $x_1^*, \ldots, x_n^*$ in E^* by $x_i^* = u^*(e_i)$, where $u^* \in \mathcal{L}(\mathbb{R}^n, E)$ $(\mathcal{L}(\mathbb{R}^n, E) = \mathcal{L}((\mathbb{R}^n)^*, E)$ since $(\mathbb{R}^n)^* = \mathbb{R}^n)$.
Then $< v, u >= tr(vu) = \sum_{i=1}^n x_i^*(x_i)$.
Let α be a norm on $\mathcal{L}(\mathbb{R}^n, E)$. On $\mathcal{L}(E, \mathbb{R}^n)$ we introduce the dual norm α^* as follows:

$$(\forall v : E \to \mathbb{R}^n)\, \alpha^*(v) = \sup\{tr(vT) \mid T : \mathbb{R}^n \to E \text{ and } \alpha(T) \leq 1\}$$

3. Lewis' Theorem

Lemma 1. *If* $f(t) \leq g(t)$ *and* $f(0) = g(0)$*, then* $f'(0) \leq g'(0)$.

Proof. Easy. □

Lemma 2. *For any* $n \times n$ *matrix* A $\quad \frac{d}{dt} \det(I_n + tA)\Big|_{t=0} = trA$

Proof.

$$\begin{aligned} det(A) &= \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^n a_{i\sigma(i)} = \\ &= \prod_{i=1}^n a_{ii} + \sum_{\sigma \neq Id} sgn(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \end{aligned}$$

$$det(I_n + tA) = \prod_{i=1}^n (1 + ta_{ii}) + \sum_{\sigma \neq Id} sgn(\sigma) \prod_{i=1}^n (I_n + tA)_{i\sigma(i)}$$

If $\sigma \neq Id$, then $|\{i \mid \sigma(i) \neq i\}| \geq 2$, but if $\sigma(i) \neq i$, then $I_{i\sigma(i)} = 0$. Hence for $\sigma \neq Id$

$$\prod_{i=1}^n (I_n + tA)_{i\sigma(i)} = \prod_{i \neq \sigma(i)} (tA)_{i\sigma(i)} \cdot \prod_{i = \sigma(i)} (I_n + tA)_{i\sigma(i)}$$

The first product on the right hand side has at least two factors, so it is divisible by t^2. Therefore $\prod_{i=1}^n (I_n + tA)_{i\sigma(i)}$ for $\sigma \neq Id$ is a polynomial in t with 2 being the lowest power of t, so its derivative is a polynomial with the lowest power of t equal 1 and at $t = 0$ the derivative is 0. Hence we get

$$\frac{d}{dt} \det(I_n + tA)\Big|_{t=0} = \sum_{i=1}^n a_{ii} \cdot \prod_{i=1}^n 1 + 0 = tr(A)$$

□

Theorem 1. *(Lewis' Theorem - generalization of John's Theorem) Let* E *be a vector space of dimension* n *and let* α *be a norm on* $\mathcal{L}(\mathbb{R}^n, E)$*. Then there is an isomorphism* $u : \mathbb{R}^n \to E$ *such that:*

$$\alpha(u) = 1 \qquad \textit{and} \qquad \alpha^*(u^{-1}) = n$$

Proof. Fix a linear basis of E and associate to it a determinant function $u \mapsto \det(u)$. Let $K = \{u \in \mathcal{L}(\mathbb{R}^n, E) \mid \alpha(u) \leq 1\}$.
K is a bounded and closed subset of the finite dimensional space $\mathcal{L}(\mathbb{R}^n, E)$, hence K is compact. Therefore the determinant, as a continuous function, attains its supremum on K. Let u be such that $|\det(u)| = \sup\{|\det(v)| \mid v \in K\}$, then for all $T \in \mathcal{L}(\mathbb{R}^n, E)$ we get $\frac{u+T}{\alpha(u+T)} \in K$ and $\left|\det\left(\frac{u+T}{\alpha(u+T)}\right)\right| \leq |\det(u)|$.
By homogeneity $|\det(u+T)| \leq |\det(u)|(\alpha(u+T))^n$. Since $\alpha(id) \neq 0$, K contains nonzero elements. Hence $\det(u) \neq 0$ and so u is invertible, $\det(u^{-1}) = \frac{1}{\det(u)}$. Then

$$\frac{|\det(u+T)|}{|\det(u)|} \leq (\alpha(u+T))^n$$

$$|\det(u^{-1}(u+T))| \leq (\alpha(u+T))^n$$

$$|\det(1+u^{-1}T)| \leq (\alpha(u+T))^n \leq (\alpha(u)+\alpha(T))^n \leq (1+\alpha(T))^n$$

Assume $\det(1+u^{-1}T) \geq 0$ and set $f(t) = \det(1+tu^{-1}T)$, $g(t) = (1+t\alpha(T))^n$. Then $f(0) = g(0) = 1$. Observe that by Lemma 2 $f'(0) = tr(u^{-1}T)$. We also have $g'(t) = n(1+t\alpha(T))^{n-1}\alpha(T)$, so $g'(0) = n\alpha(T)$.
Hence by Lemma 1 $tr(u^{-1}T) \leq n\alpha(T)$ and this inequality holds for all $T : \mathbb{R}^n \to E$. Since $u^{-1} : E \to \mathbb{R}^n$ we get:

$$\alpha^*(u^{-1}) = \sup\{tr(u^{-1}T) \mid T : \mathbb{R}^n \to E \text{ and } \alpha(T) \leq 1\} \leq n \cdot 1 = n$$

On the other hand $u^{-1}u = I_n$, so $n = tr(I_n) = tr(u^{-1}u) \leq \alpha(u)\alpha^*(u^{-1})$ by the definition of α^*.
So $n \leq \alpha(u)\alpha^*(u^{-1}) \leq \alpha(u)n \Rightarrow \alpha(u) = 1$ and $\alpha^*(u^{-1}) = n$. □

Remark. The theorem can be reformulated as follows:
Since $u \mapsto (u(e_1), \ldots, u(e_n))$ gives a correspondence between $\mathcal{L}(\mathbb{R}^n, E)$ and E^n, then let α be a norm on E^n. Then there is a linear basis $(x_1, \ldots, x_n)$ of E such that the biorthogonal functionals $(x_1^*, \ldots, x_n^*)$ satisfy

$$\alpha((x_1, \ldots, x_n))\alpha^*((x_1^*, \ldots, x_n^*)) = n$$

In particular for $\alpha((x_1, \ldots, x_n)) = 1$ we get $\alpha^*((x_1^*, \ldots, x_n^*)) = n$.

Corollary 1. *(Auerbach Lemma) Let E be a normed space of dimension n. There is a basis $(x_1, \ldots, x_n)$ of E such that*

$$\forall(\alpha_i) \in \mathbb{R}^n \qquad \sup|\alpha_i| \leq \left\|\sum_{i=1}^n \alpha_i x_i\right\| \leq \sum_{i=1}^n |\alpha_i|$$

Proof. Set $\alpha((x_1, \ldots, x_n)) = \sup\|x_i\| = \max_{i \leq i \leq n}\|x_i\|_E$ - norm on $l_\infty^n(E)$. Then by the Remark there exists a basis $(x_1, \ldots, x_n)$ in E such that biorthogonal functionals $(x_1^*, \ldots, x_n^*)$ satisfy

$$\max_{i \leq i \leq n}\|x_i\|_E = 1 \qquad \text{and} \qquad \sum_{i=1}^n \|x_i^*\| = n$$

because $l_1^n(E^*)$ is the dual of $l_\infty^n(E)$ and $\alpha^*((x_1^*, \ldots, x_n^*)) = \sum_{i=1}^n \|x_i^*\|$.
Since $x_i^*(x_i) = 1$ and $\max_{i \leq i \leq n}\|x_i\|_E = 1$, we have $\|x_i^*\| \geq 1$, $i = 1, \ldots, n$.
Therefore $\sum_{i=1}^n \|x_i^*\| = n$ yields $\|x_i^*\| = 1 \quad \forall i = 1, \ldots, n$.

Observe that $\alpha_i = x_i^* \left(\sum_{j=1}^n \alpha_j x_j\right)$, so

$$\sup|\alpha_i| = \sup\left|x_i^* \left(\sum_{j=1}^n \alpha_j x_j\right)\right| \leq \sup \|x_i^*\| \cdot \left\|\sum_{j=1}^n \alpha_j x_j\right\| = \left\|\sum_{j=1}^n \alpha_j x_j\right\| \leq$$

$$\leq \sum_{j=1}^n |\alpha_j| \cdot \|x_j\| \leq \max_i \|x_i\| \cdot \sum_{j=1}^n |\alpha_j| = \sum_{j=1}^n |\alpha_j|$$

$\square$

References

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