

MATH438 - TOPOLOGICAL DYNAMICS - FALL 03
POINCARÉ RECURRENCE THEOREM

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1. SOME FACTS FROM MEASURE THEORY

Here X is a Hausdorff space. Let Σ be the Borel σ -algebra of subsets of X , i.e. the smallest σ -algebra containing all open subsets of X .

- A *Borel measure* is a measure μ , defined on Σ , such that $\mu(F) < \infty$ for every compact $F \in X$.
- A Borel measure μ is *inner regular* if $\mu(U) = \sup\{\mu(F) \mid F \in U \text{ and } F \text{ compact}\} \forall U - \text{open in } X$.
- A Borel measure μ is *outer regular* if $\mu(B) = \inf\{\mu(U) \mid B \in U \text{ and } U \text{ open}\} \forall B \in \Sigma$.
- A Borel measure is *regular* if it is both inner and outer regular. An example of a Borel measure is the Lebesgue measure.
- A *probability measure* on X is a regular Borel measure μ on Σ such that $\mu(X) = 1$. The set of all probability measures on X is denoted by $M_1(X)$.
- Let μ be an inner regular, finite Borel measure on X . Then there exists a smallest closed subset S of X with the property that $\mu(S) = \mu(X)$. The set S is called *the support* of μ , denoted $\text{supp } \mu$.

Note. \mathfrak{X} denotes here either a continuous flow $\langle \mathbb{R}, X, \pi \rangle$ or a discrete flow (\mathbb{Z} -flow) $\langle X, h \rangle$. μ denotes a regular Borel measure on X .

Definition 1. A measure μ on X is invariant if for every set $A \in \Sigma$

- (a) $\mu(\pi_{-t}(A)) = \mu(A) \forall t \in \mathbb{R}$ in case of $\langle \mathbb{R}, X, \pi \rangle$,
- (b) $\mu(h^{-1}(A)) = \mu(A)$ in case of $\langle X, h \rangle$.

If μ is an invariant measure for \mathfrak{X} , we shall also say that μ is preserved by the transformations in \mathfrak{X} (i.e. by π_t or h), or that the transformations in \mathfrak{X} preserve μ .

Remarks.

- (i) An invariant (probability) measure for the continuous flow $\langle \mathbb{R}, X, \pi \rangle$ is also an invariant (probability) measure for the associated discrete flow $\langle X, \pi_1 \rangle$.
- (ii) Let μ be an invariant probability measure for the flow \mathfrak{X} . Then the $\text{supp } \mu$ is a closed invariant subset of X .

From measure-theoretic point of view the subflow on $\text{supp } \mu$ contains all relevant information. The support of μ is equal to the phase space of the subflow, i.e. every non-empty open subset has positive measure.

Fact. Every continuous or discrete flow on a compact space has an invariant probability measure.

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Proof. It is a consequence of Markov-Kakutani Fixed Point Theorem. \square

Example 1. $\mathfrak{X} = \langle X, h \rangle$, where $X = [0, 1]$, $h(x) = x^2$.

Claim: The point measures δ_0 and δ_1 are invariant.

Proof. Let μ be any invariant probability measure on X . Take $0 < a < b < 1$ and $n \in \mathbb{Z}$ such that $a^{2^n} > b$. Then $\mu(a, 1] = \mu(f^n(a, 1]) = \mu(a^{2^n}, 1]$. Hence $\mu(a, b) = 0$ and therefore $\text{supp } \mu \subset \{0, 1\}$. So there exist $0 \leq r, s \leq 1$ such that $\mu(\{0\}) = r$, $\mu(\{1\}) = s$ and $r + s = 1$.

Hence $\mu = \lambda\delta_0 + (1 - \lambda)\delta_1$ for some $\lambda \in [0, 1]$.

For $\lambda = 1$ we get δ_0 invariant, for $\lambda = 0$ we get δ_1 invariant. \square

Definition 2. Let \mathfrak{X} be either a continuous or a discrete flow. $x \in X$ is recurrent if:

$$(\forall U \in \mathcal{N}_x) \quad D(x, U) := \{t \mid tx \in U\}$$

is neither bounded from above nor from below.

Recall the definition of almost periodic point: x is AP if $(\forall U \in \mathcal{N}_x) D(x, U)$ is syndetic.

Definition 3. If \mathfrak{X} is a discrete flow $\langle X, h \rangle$, then $x \in X$ is

- (a) positively recurrent if $(\forall U \in \mathcal{N}_x) \quad D(x, U) = \{n \in \mathbb{N} \mid h^n(x) \in U\}$ is unbounded,
- (b) negatively recurrent if $(\forall U \in \mathcal{N}_x) \quad D(x, U) = \{n \leq 0 \mid h^n(x) \in U\}$ is unbounded.

2. POINCARÉ RECURRENCE THEOREM

Let (X, \mathcal{B}, μ) be a probability space, $f : X \rightarrow X$ a transformation measurable with respect to \mathcal{B} , i.e. $(\forall B \in \mathcal{B}) f^{-1}(B) \in \mathcal{B}$, such that f preserves μ (f does not have to be invertible).

Theorem 1. (*Poincaré Recurrence Theorem*) Let $B \in \mathcal{B}$ be such that $\mu(B) > 0$. Then almost all points of B return infinitely often to B , i.e. for almost all $x \in B$ the set

$$D(x, B) = \{n \in \mathbb{N} \mid f^n(x) \in B\}$$

is infinite (unbounded).

(almost all = all except for a set of measure zero)

Proof. The set of all points in B which never return to B is $N_1 = B \setminus \bigcup_{n=1}^{\infty} f^{-n}(B)$. Since f is measurable, N_1 belongs to \mathcal{B} . We will show that $\mu(N_1) = 0$.

Claim. For $i \neq j \quad f^{-i}(N_1) \cap f^{-j}(N_1) = \emptyset$.

Assume that for some $0 \leq i < j$ we have $f^{-i}(N_1) \cap f^{-j}(N_1) \neq \emptyset$, then for some $y \in N_1 \subset B$ we get that $f^{-i+j}(y) \in N_1 \subset B$, contradicting the definition of N_1 .

Since f preserves measure μ , all the sets $f^{-i}(N_1)$, $i = 0, 1, 2, \dots$, have the same measure. Hence

$$\sum_{i=0}^{\infty} \mu(N_1) = \sum_{i=0}^{\infty} \mu(f^{-i}(N_1)) = \mu\left(\bigcup_{i=1}^{\infty} (f^{-i}(N_1))\right) \leq \mu(X) = 1$$

Therefore $\sum_{i=0}^{\infty} \mu(N_1) \leq 1 \implies \mu(N_1) = 0$.

Now consider f^p in place of f and the set N_p in place of N_1 , for some $p \in \mathbb{N}$. $\mu(N_p) = 0$ and $\forall x \in B \setminus N_p$ there exists $n \in \mathbb{N}$ such that $f^{pn}(x) \in B$. Hence

$D(x, B)$ contains an element $m = np \geq p$.

If $x \in \bigcap_{p=1}^{\infty} (B \setminus N_p) = B \setminus \bigcup_{p=1}^{\infty} N_p$, then $D(x, B)$ contains arbitrarily large elements ($\geq p$ for all p).

Then $\mu(N_p) = 0 \implies \mu\left(\bigcup_{p=1}^{\infty} N_p\right) = 0 \implies$ for almost all $x \in B$ the set $D(x, B)$ is unbounded. \square

Corollary 1. *Assume that a (continuous or discrete) flow \mathfrak{X} has an invariant probability measure. If, in addition, X has a countable base for its topology, then almost all points in X are recurrent.*

Proof. Case 1: Consider a discrete flow $\langle X, h \rangle$.

Let $\{U_n\}_{n=1}^{\infty}$ be a base for X . $U_n \in \mathcal{B}$, so by Poincaré Recurrence Theorem ($\forall i \in \mathbb{N}$) (\exists null-set $N_i \subset U_i$) ($\forall x \in U_i \setminus N_i$) $D(x, U_i)$ is not bounded from above. Then $N^+ := \bigcup_{i=1}^{\infty} N_i$ is a null-set and every point in $X \setminus N^+$ is positively recurrent, because:

if $x \in X \setminus N^+$, then every neighborhood of x contains (at least) one of the sets $U_i \setminus N_i$, and x must belong to $U_i \setminus N_i$ (since $x \notin \bigcup_{i=1}^{\infty} N_i$).

Similarly, one finds the null-set N^- such that every point in $X \setminus N^-$ is positively recurrent under h^{-1} , i.e. negatively recurrent under h .

Now $N := N^+ \cup N^-$ is a null-set, and each point of $X \setminus N$ is recurrent.

Case 2: Continuous flow.

We need the following lemma:

Lemma 1. $\mathfrak{X} = \langle \mathbb{R}, X, \pi \rangle$, $\mathfrak{X}_{\mathbb{Z}} = \langle \mathbb{Z}, X, \pi|_{\mathbb{Z} \times X} \rangle = \langle X, \pi_1 \rangle$, then:
 $x \in X$ is recurrent in $\mathfrak{X}_{\mathbb{Z}} \implies x \in X$ is recurrent in \mathfrak{X} .

We shall apply Case 1 to $\langle X, \pi_1 \rangle$ to get the result for $\mathfrak{X}_{\mathbb{Z}}$. Then by Lemma we get the result for \mathfrak{X} . \square

REFERENCES

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