TRACE CLASS

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Let H be a Hilbert space and let $A \in B(H)$. Let also $\{e_i\}$ and $\{f_i\}$ be two orthonormal bases of H. Our first observation is:

$$\sum_i \|Ae_i\|^2 = \sum_i \|Af_i\|^2 = \sum_i \sum_j |\langle Ae_i, f_j \rangle|^2$$

Proof.

$$\begin{aligned}\sum_i \|Ae_i\|^2 &= \sum_i \left\| \sum_j \langle Ae_i, f_j \rangle f_j \right\|^2 \\ &= \sum_i \left\langle \sum_j \langle Ae_i, f_j \rangle f_j, \sum_k \langle Ae_i, f_k \rangle f_k \right\rangle \\ &= \sum_i \sum_j \langle \langle Ae_i, f_j \rangle f_j, \langle Ae_i, f_j \rangle f_j \rangle = \sum_i \sum_j |\langle Ae_i, f_j \rangle|^2\end{aligned}$$

Now, since $1 = \langle e_i, e_i \rangle = \langle e_i, f_j \rangle \langle f_j, e_i \rangle = |\langle e_i, f_j \rangle|^2$, we get

$$\langle Ae_i, f_j \rangle = \langle e_i, f_j \rangle \langle Af_j, \langle f_j, e_i \rangle e_i \rangle = |\langle e_i, f_j \rangle|^2 \langle Af_j, e_i \rangle = \langle Af_j, e_i \rangle$$

so $\sum_i \sum_j |\langle Ae_i, f_j \rangle|^2 = \sum_j \sum_i |\langle Af_j, e_i \rangle|^2 =$ ...(manipulations as above)... $= \sum_j \|Af_j\|^2$. □

Therefore the expression $\sum_i \|Ae_i\|^2$ does not depend on the choice of a basis and hence we can define:

$$\|A\|_2 = \left(\sum_i \|Ae_i\|^2 \right)^{1/2}$$

and a new space of so called Hilbert - Schmidt operators:

$$S_2(H) = \{A \in B(H) \mid \|A\|_2 < \infty\}$$

We still need to prove that $\|\cdot\|_2$ is a norm on $S_2(H)$.

Proof.

(i) Assume $\|A\|_2 = 0$. Then for all i $\|Ae_i\| = 0$, and since A is 0 on all basis vectors, it is 0 itself.

(ii) Homogenity is clear.

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(iii)

$$\begin{aligned}
\|A+B\|_2^2 &= \sum_i \|(A+B)e_i\|^2 = \sum_i \|Ae_i + Be_i\|^2 \\
&\leq \sum_i (\|Ae_i\| + \|Be_i\|)^2 = \sum_i \left(\|Ae_i\|^2 + 2\,\|Ae_i\|\,\|Be_i\| + \|Be_i\|^2\right) \\
&= \sum_i \|Ae_i\|^2 + 2\sum_i \|Ae_i\|\,\|Be_i\| + \sum_i \|Be_i\|^2 \\
&\leq \left[\left(\sum_i \|Ae_i\|^2\right)^{1/2}\right]^2 + 2\left(\sum_i \|Ae_i\|^2\right)^{1/2}\left(\sum_i \|Be_i\|^2\right)^{1/2} + \left[\left(\sum_i \|Be_i\|^2\right)^{1/2}\right]^2 \\
&= \left[\left(\sum_i \|Ae_i\|^2\right)^{1/2} + \left(\sum_i \|Be_i\|^2\right)^{1/2}\right]^2
\end{aligned}$$

hence $\|A+B\|_2 \leq \|A\|_2 + \|B\|_2$.

□

Proposition 1. For $A \in S_2(H)$ holds $\|A\| \leq \|A\|_2$.

Proof. $\|A\|_2^2 = \sum_i \langle Ae_i, Ae_i\rangle = \sum_i \langle A^*Ae_i, e_i\rangle \geq \sup\{\langle A^*Ah, h\rangle \mid \|h\| \leq 1\} = \|A^*A\| = \|A\|^2$ □

Proposition 2. If $T \in B(H)$, $A \in S_2(H)$, then $\|TA\|_2 \leq \|T\|\,\|A\|_2$

Proof. $\|TA\|_2^2 = \sum_i \|TAe_i\|^2 \leq \sum_i \|T\|^2\,\|Ae_i\|^2 = \|T\|^2 \sum_i \|Ae_i\|^2$ □

Proposition 3. For $A \in S_2(H)$, $\|A^*\|_2 = \|A\|_2$.

Proof.

$$\begin{aligned}
\|A^*\|_2^2 &= \sum_i \|A^*e_i\|^2 = \sum_i \sum_j |\langle A^*e_i, f_j\rangle|^2 \\
&= \sum_i \sum_j |\langle e_i, Af_j\rangle|^2 = \sum_j \sum_i |\langle Af_j, e_i\rangle|^2 = \sum_j \|Af_j\|^2 = \|A\|_2^2
\end{aligned}$$

□

Proposition 4. If $T \in B(H)$, $A \in S_2(H)$, then $\|AT\|_2 \leq \|T\|\,\|A\|_2$

Proof. $\|AT\|_2 = \|(AT)^*\|_2 = \|T^*A^*\|_2 \leq \|T^*\|\,\|A^*\|_2 = \|T\|\,\|A\|_2$. □

We can now conclude that $S_2(H)$ is a two-sided ideal of $B(H)$. Also, $S_2(H)$ contains finite rank operators.(since the sum in the norm is finite...)

Proposition 5. $A \in S_2(H) \Rightarrow |A| := (A^*A)^{1/2} \in S_2(H)$.

Proof.

$$\begin{aligned}\sum_i \left\|(A^*A)^{1/2}e_i\right\|^2 &= \sum_i \left\langle (A^*A)^{1/2}e_i, (A^*A)^{1/2}e_i \right\rangle \\ &= \sum_i \left\langle (A^*A)^{1/2}(A^*A)^{1/2}e_i, e_i \right\rangle = \sum_i \langle A^*Ae_i, e_i\rangle \\ &= \sum_i \langle Ae_i, Ae_i\rangle = \sum_i \|Ae_i\|^2\end{aligned}$$

So $\||A|\|_2 = \|A\|_2$. □

Proposition 6. $S_2(H)$ is contained in $K(H)$ - compact operators on H.

Proof. Let $\{e_i \mid i \in I\}$ be ONB of H and let $\varepsilon > 0$. Consider $T \in S_2(H)$ then $\sum_i \|Te_i\|^2 < \infty$, so $\sum_i \left\langle |T|^2 e_i, e_i \right\rangle < \infty$ and there exists a finite set $J \subset I$ such that $\sum_{i \notin J} \left\langle |T|^2 e_i, e_i \right\rangle < \varepsilon$. Let P_J be a projection onto a span$\{e_i \mid i \in J\}$, P_J is a finite-rank operator. Then:

$$\begin{aligned}\||T|(1-P_J)\|_2^2 &= \sum_i \||T|(1-P_J)e_i\|^2 = \sum_{i \notin J} \||T|e_i\|^2 \\ &= \sum_{i \notin J} \langle |T|e_i, |T|e_i\rangle = \sum_{i \notin J} \left\langle |T|^2 e_i, e_i\right\rangle < \varepsilon\end{aligned}$$

Since ε is arbitrary, $|T| \in \overline{B_f(H)}^{\|\cdot\|}$, so $|T|$ is compact. $T = U|T|$, so T is also compact.

This also shows that finite rank operators are dense in $S_2(H)$ since

$$S_2(H) \subset K(H) = \overline{B_f(H)}^{\|\cdot\|}$$

Another way to prove this proposition is to use the fact, that an operator T is compact if and only if for any ONB $\{e_i\}$ holds $\|Te_i\| \xrightarrow[i \to 0]{} 0$. If $T \in S_2(H)$, then $\sum_i \|Te_i\| < \infty$, so in particular $\|Te_i\| \xrightarrow[i \to 0]{} 0$, hence $T \in K(H)$. □

Definition 7. The trace-class operators $S_1(H) := \{AB \mid A, B \in S_2(H)\}$.

Proposition 8. $A \in S_1(H)$ if and only if $|A| \in S_1(H)$.

Proof. It follows from the fact that $S_2(H)$ is a two-sided ideal.

If $|A| \in S_2(H)$ then $A = U|A| = UBC$ with U - partial isometry, $B, C \in S_2(H)$, so $UB \in S_2(H)$ and $A \in S_1(H)$.

If $A \in S_2(H)$ then $|A| = U^*A = U^*EF$ with U - partial isometry, $E, F \in S_2(H)$, so $U^*E \in S_2(H)$ and $|A| \in S_1(H)$. □

Proposition 9.

(1) If $A \in S_1(H)$, $\{e_i\}$ is an o-n system of H, then $\sum_i |\langle Ae_i, e_i\rangle| < \infty$.
(2) If $A \in S_1(H)$, $\{e_i\}$ and $\{f_i\}$ are o-n systems of H, then $\sum_i |\langle Ae_i, f_i\rangle| < \infty$.

Proof.

(1) So let $A = BC$ with $B, C \in S_2(H)$.

$$\sum_i |\langle Ae_i, e_i\rangle| = \sum_i |\langle BCe_i, e_i\rangle| = \sum_i |\langle Ce_i, B^* e_i\rangle|$$
$$\leq \sum_i \|Ce_i\| \|B^* e_i\| \leq \left(\sum_i \|Ce_i\|^2\right)^{1/2} \left(\sum_i \|B^* e_i\|^2\right)^{1/2} \leq \|C\|_2 \|B\|_2 < \infty$$

(2) So let $A = BC$ with $B, C \in S_2(H)$.

$$\sum_i |\langle Ae_i, f_i\rangle| = \sum_i |\langle BCe_i, f_i\rangle| = \sum_i |\langle Ce_i, B^* f_i\rangle|$$
$$\leq \sum_i \|Ce_i\| \|B^* f_i\| \leq \left(\sum_i \|Ce_i\|^2\right)^{1/2} \left(\sum_i \|B^* f_i\|^2\right)^{1/2} \leq \|C\|_2 \|B\|_2 < \infty$$

□

Definition 10. If $\{e_i\}$ is a basis of H, define the trace to be $tr : S_1(H) \to \mathbb{C}$, $tr(A) = \sum_i \langle Ae_i, e_i\rangle$.

Example 11. $A \in M_n$, $A = [a_{ij}]_{n\times n}$, $tr(A) = \sum_{i=1}^{n} a_{ii}$

$tr : S_1(H) \to \mathbb{C}$ is a positive faithful linear functional:
$A \geq 0 \Rightarrow \langle Ae_i, e_i\rangle \geq 0$ for all i and hence $tr(A) \geq 0$
$A \geq 0$ and $tr(A) = 0 \Rightarrow \langle Ae_i, e_i\rangle = 0$ for all i, since $\{e_i\}$ is a basis we get $\langle Ah, h\rangle = 0$ for all $h \in H$ and $A = 0$.

Proposition 12. Let $A \in B(H)$. Then $tr(|A|) < \infty$ iff $|A| \in S_1(H)$.

Proof. ($\Rightarrow$)

$\sum_i \left\| |A|^{1/2} e_i \right\|^2 = \sum_i \left\langle |A|^{1/2} e_i, |A|^{1/2} e_i \right\rangle = \sum_i \langle |A| e_i, e_i\rangle = tr(|A|) < \infty$. So $|A|^{1/2} \in S_2(H)$. Therefore $|A| = |A|^{1/2} |A|^{1/2} \in S_1(H)$.

($\Leftarrow$)

$|A| \in S_1(H)$, so $|A| = DC$ with $D, C \in S_2(H)$ and

$$tr(|A|) = \sum_i \langle DCe_i, e_i\rangle = \sum_i \langle Ce_i, D^* e_i\rangle \leq \sum_i \|Ce_i\| \|D^* e_i\|$$
$$\leq \left(\sum_i \|Ce_i\|^2\right)^{1/2} \left(\sum_i \|D^* e_i\|^2\right)^{1/2} = \|C\|_2 \|D^*\|_2 < \infty$$

□

Proposition 13. For $A \in S_1(H)$ and $T \in B(H)$ hold: $AT \in S_1(H)$, $TA \in S_1(H)$ and $tr(AT) = tr(TA)$.

Proof. $A \in S_1(H)$, so $A = BC$ for $B, C \in S_2(H)$. Since $S_2(H)$ is an ideal, we get: $AT = BCT = B(CT) \in S_1(H)$, $TA = TBC = (TB)C \in S_1(H)$.

$$\begin{aligned} tr(AT) &= \sum_i \langle ATe_i, e_i \rangle = \sum_i \langle Te_i, A^* e_i \rangle = \sum_i \sum_j \langle Te_i, e_j \rangle \langle e_j, A^* e_i \rangle \\ &= \sum_j \sum_i \langle e_i, T^* e_j \rangle \langle Ae_j, e_i \rangle = \sum_j \sum_i \langle Ae_j, e_i \rangle \langle e_i, T^* e_j \rangle \\ &= \sum_j \langle Ae_j, T^* e_j \rangle = \sum_j \langle TAe_j, e_j \rangle = tr(TA) \end{aligned}$$

□

We conclude that $S_1(H)$ is a two-sided ideal in $B(H)$ and it contains finite rank operators.

Let U be a unitary operator in $B(H)$, and let $T \in S_1(H)$. Then by proposition 13:

$$tr(UTU^*) = tr(U^*UT) = tr(T)$$

Therefore the trace is independent of the choice of basis of H.

Proposition 14. $S_2(H)$ is a Hilbert space with the inner product

$$\langle A, B \rangle_{tr} := tr(B^* A)$$

Proof. check if it is inner product - easy.

Is $\|\cdot\|_2$ comming from $\langle \cdot, \cdot \rangle_{tr}$?

$$\begin{aligned} \langle A, A \rangle_{tr} = tr(A^* A) &= \sum_i \langle A^* A e_i, e_i \rangle = \sum_i \left\langle |A|^2 e_i, e_i \right\rangle \\ &= \sum_i \langle |A| e_i, |A| e_i \rangle = \sum_i \| |A| e_i \| = \| |A| \|_2^2 = \|A\|_2^2 \end{aligned}$$

Hence $\|A\|_2 = \sqrt{\langle A, A \rangle_{tr}}$ indeed.

Completness: Recall that $\|A\| \le \|A\|_2$. So if $\{A_n\}$ is a Cauchy sequence in $S_2(H)$, it is also a Cauchy sequence in the operator norm, and since $B(H)$ is complete, there exists $T \in B(H)$ such that $\|T_n - T\| \to 0$. Let P be a projection on a finite dimensional subspace of H:

$$\begin{aligned} \|P(T_n - T)\|_2^2 &= tr((T_n - T)^* P (T_n - T)) = tr(P(T_n - T)(T_n - T)^* P) \\ &= \lim_m tr(P(T_n - T_m)(T_n - T_m)^* P) = \lim_m tr((T_n - T_m)^* P (T_n - T_m)) \\ &\le \limsup_m tr((T_n - T_m)^* P (T_n - T_m)) = \limsup_m \|T_m - T_n\|_2^2 \end{aligned}$$

Since P is arbitrary, $\|T - T_n\|_2 \le \limsup_m \|T_m - T_n\|_2$, so $\|T - T_n\|_2 \to 0$. Also, $\|T\|_2 \le \|T - T_n\|_2 + \|T_n\|_2 < \infty$, hence $T \in S_2(H)$. □

Proposition 15. If $T \in S_1(H)$ and $S \in B(H)$ then $|tr(ST)| \le \|S\| \, tr(|T|)$.

Proof. Write $T = U|T|$, hence $|T| = U^*T$, and $|T|^{1/2} \in S_2(H)$. Then $SU|T|^{1/2} \in S_2(H)$ and $(SU|T|^{1/2})^* \in S_2(H)$.

$$\begin{aligned}
|tr(ST)|^2 = |tr(SU|T|)|^2 &= \left|tr(SU|T|^{1/2}|T|^{1/2})\right|^2 \\
&= \left|\left\langle (SU|T|^{1/2})^*, |T|^{1/2}\right\rangle_{tr}\right|^2 \le \left\|(SU|T|^{1/2})^*\right\|_2^2 \left\||T|^{1/2}\right\|_2^2 \\
&= tr(SU|T|^{1/2}|T|^{1/2}U^*S^*)tr(|T|) = tr(|T|)tr(U^*S^*SU|T|) \\
&\le tr(|T|)tr(\|U^*S^*SU\|1|T|) = tr(|T|)^2\|SU\|^2 \le tr(|T|)^2\|S\|^2
\end{aligned}$$

then $|tr(ST)| \le tr(|T|)\|S\|$. ☐

Definition 16. For $A \in S_1(H)$ define $\|A\|_1 := tr(|A|)$, the trace norm.

We first need to show that trace norm is a norm on $S_1(H)$:

(i) $\|A\|_1 = 0 \iff \sum_i \underbrace{\langle |A| e_i, e_i\rangle}_{\ge 0} = 0 \iff (\forall i)\ \langle |A| e_i, e_i\rangle = 0 \iff (\forall h \in H)$ $\langle |A| h, h\rangle = 0 \iff |A| = 0$. Then since $A = U|A|$, where U is a nonzero partial isometry, we get $A = 0$.

(ii) homogenity is clear

(iii) Consider S and T from $S_1(H)$. Let U be the partial isometry such that $S + T = U|S+T|$. Then using proposition 15:

$$\begin{aligned}
\|S+T\|_1 &= tr(U^*(S+T)) = tr(U^*S + U^*T) \\
&= tr(U^*S) + tr(U^*T) \le |tr(U^*S)| + |tr(U^*T)| \\
&\le \|U^*\|tr(|S|) + \|U^*\|tr(|T|) \le tr(|S|) + tr(|T|) = \|S\|_1 + \|T\|_1
\end{aligned}$$

Proposition 17. $\|T\| \le tr(|T|) = \|T\|_1$

Proof. If T is normal, then $\|T\| = \sup\{\langle Tx, x\rangle \mid \|x\| \le 1\}$. Assume $T \ge 0$, then $\|T\| = \sup\{\langle Tx, x\rangle \mid \|x\| \le 1\} \le \sum_i \langle Te_i, e_i\rangle = tr(T)$. For any T we get
$\|T\| = \|T^+ - T^-\| \le \|T^+\| + \|T^-\| \le tr(T^+) + tr(T^-) = tr(|T|)$ ☐

Proposition 18. If $T \in S_1(H)$ then $\|T\|_1 = \sup\{|tr(CT)| \mid \|C\| \le 1, C \in K(H)\}$.

Proof. For any $C \in K(H)$ $|tr(CT)| \le \|C\|\|T\|_1$, so $\sup\limits_{\|C\|\le 1} |tr(CT)| \le \|T\|_1$

Since $|T| = UT$ with $\|U\| \le 1$, we get $\|T\|_1 = tr(|T|) = tr(UT) \le \sup\limits_{\|C\|\le 1} |tr(CT)|$ ☐

Proposition 19. If $A \in S_1(H)$ and $T \in B(H)$, then:

(a) $|tr(TA)| \le \|T\|\|A\|_1$
(b) $\|A^*\|_1 = \|A\|_1$
(c) $\|TA\|_1 \le \|T\|\|A\|_1$
(d) $\|AT\|_1 \le \|T\|\|A\|_1$

Proof.

(a) This is proposition 15.

(b) $\| |A| \|_1 = tr(|A|) = \|A\|_1$ and $\|A\|_1^2 = \||A|\|_1^2 = \|A^*A\|_1$

$$\|A^*\|_1^2 = \||A^*|\|_1^2 = \|AA^*\|_1 = tr(AA^*) = tr(A^*A) = \|A^*A\|_1 = \|A\|_1^2$$

(c)

$$\|TA\|_1 = \sup_{\|C\|\leq 1} |tr(CTA)| \leq \sup_{\|C\|\leq 1} \|CT\| \|A\|_1 \leq \sup_{\|C\|\leq 1} \|C\| \|T\| \|A\|_1 \leq \|T\| \|A\|_1$$

(d) $\|AT\|_1 = \|T^*A^*\|_1 \leq \|T^*\| \|A^*\|_1 = \|T\| \|A\|_1$

□

Proposition 20. $S_1(H) \subset K(H)$

Proof. If $T \in S_1(H)$, then $T = AB$ with $A, B \in S_2(H) \subset K(H)$. Since $K(H)$ is an ideal itself, AB is compact, hence $T \in K(H)$. □

Proposition 21. $S_1(H) \subset S_2(H)$

Proof. Observe that $\|T\|_2^2 = \sum_i \langle Te_i, Te_i\rangle = \sum_i \langle T^*Te_i, e_i\rangle = tr(T^*T)$. If $T \in S_1(H)$, then $tr(|T|) < \infty$, so $|T| \in S_1(H)$, and also $|T|^2 \in S_1(H)$ since $S_1(H)$ is an ideal. Therefore $\|T\|_2^2 = tr(T^*T) = tr(|T|^2) < \infty$, so $T \in S_2(H)$. □

Proposition 22. $(S_1(H), \|\cdot\|_1)$ is a Banach space.

Proof. Let $\{T_n\}$ be a $\|\cdot\|_1$ - Cauchy sequence in $S_1(H)$. Since $\|\cdot\|_1 \geq \|\cdot\|$, $\|T_n - T\| \to 0$ for some $T \in B(H)$. Let P be a finite rank projection and U the partial isometry such that $U^*(T - T_n) = |T - T_n|$:

$$tr(PU^*(T - T_n)) = \lim_m tr(PU^*(T_m - T_n)) \leq \limsup_m \|PU^*\| tr(|T_m - T_n|)$$

so $tr(P|T - T_n|) = tr(PU^*(T - T_n)) \leq \limsup_m \|T_m - T_n\|_1$.
P - arbitrary implies that $\|T - T_n\|_1 \leq \limsup_m \|T_m - T_n\|_1 \to 0$ and $\|T\|_1 \leq \|T - T_n\|_1 + \|T_n\|_1 < \infty$, so $T \in S_1(H)$. □

Also $\|ST\|_1 = tr(U^*ST) \leq \|U^*S\| tr(|T|) \leq \|S\| tr(|T|) \leq \|S\|_1 \|T\|_1$, so $S_1(H)$ is a Banach algebra. And from previous proposition we also have $\|T\|_1 = \|T^*\|_1$ for all $T \in S_1(H)$, hence $S_1(H)$ is a C^*-algebra.

Theorem 23. $(K(H))^* \cong S_1(H)$

Proof. For every $T \in S_1(H)$ we can define a functional on $K(H)$: $L_T : K(H) \to \mathbb{C}$, $L_T(S) = tr(ST)$.

$$|L_T(S)| = |tr(ST)| \leq \|S\| \|T\|_1 < \infty$$

so L_T is bounded and $\|L_T\| \leq \|T\|_1$. Also

$$\|T\|_1 = tr(|T|) = tr(U^*T) = L_T(U^*) \leq \|L_T\| \|U^*\| \leq \|L_T\|$$

so $\|L_T\| = \|T\|_1$. Therefore $\psi : S_1(H) \to (K(H))^*$, $\psi(T) = L_T$, is an isometry. Clearly ψ is also linear, hence it is 1-1.

We will show that ψ is "onto".
If $f \in (K(H))^*$, let $S \in S_2(H)$, then

$$|f(S)| \leq \|f\| \|S\| \leq \|f\| \|S\|_2$$

Since $S_2(H)$ is a Hilbert space, there exists unique element T^* in $S_2(H)$ such that:

$$f(S) = \langle S, T^* \rangle_{tr} = tr(TS) = tr(ST) = L_T(S)$$

for all $S \in S_2(H)$. We need to show that T is in $S_1(H)$.
Consider a finite rank projection P, then $P\,|T|$ is a finite rank operator and $P\,|T| \in S_1(H) \subset S_2(H)$.

$$|tr(P\,|T|)| = |tr(PU^*T)| = |f(PU^*)| \leq \|f\|\,\|PU^*\|$$

Since P is arbitrary, $|T|$ is in $S_1(H)$ and $tr(|T|) \leq \|f\|$ so $T \in S_1(H)$ with $\|T\|_1 \leq \|f\|$. Therefore ψ is an isometric bijection between $S_1(H)$ and $(K(H))^*$. □

Theorem 24. $(S_1(H))^* \cong B(H)$

Proof. If $T \in B(H)$, then define $L_T : S_1(H) \to \mathbb{C}$, $L_T(A) = tr(TA)$.

$$|L_T(A)| = |tr(TA)| \leq \|T\|\,\|A\|_1$$

$\|L_T\| \leq \|T\|$, so $L_T \in (S_1(H))^*$.

For $f \in (S_1(H))^*$ define a sesquilinear form on $H \times H$:

$$B(g,h) = f(g \otimes h)$$

where $g \otimes h$ is rank-one operator on $S_1(H)$, $g \otimes h(a) = \langle h, a \rangle\, g$. Then

$$|B(g,h)| = |f(g \otimes h)| \leq \|f\|\,\|g \otimes h\|_1 = \|f\|\,tr(|g \otimes h|)$$

Observe first few facts about $g \otimes h$:

$$\langle g \otimes h(a), b \rangle = \langle \langle h, a \rangle g, b \rangle = \langle a, h \rangle \langle g, b \rangle = \langle a, \langle g, b \rangle h \rangle = \langle a, h \otimes g(b) \rangle$$

So $(g \otimes h)^* = h \otimes g$. It follows that

$$\begin{aligned} |g \otimes h|^2(a) &= (h \otimes g)(g \otimes h)(a) = (h \otimes g)\langle h, a \rangle g = \langle h, a \rangle \langle g, g \rangle h \\ &= \|g\|^2 (h \otimes h)(a) = \|g\|^2 \|h\|^2 \left(\frac{h}{\|h\|} \otimes \frac{h}{\|h\|} \right)(a) \end{aligned}$$

Let us check whether $\left(\frac{h}{\|h\|} \otimes \frac{h}{\|h\|} \right)$ is idenpotent:

$$\begin{aligned} \left(\frac{h}{\|h\|} \otimes \frac{h}{\|h\|} \right)\left(\frac{h}{\|h\|} \otimes \frac{h}{\|h\|} \right)(a) &= \left(\frac{h}{\|h\|} \otimes \frac{h}{\|h\|} \right)\left\langle \frac{h}{\|h\|}, a \right\rangle \frac{h}{\|h\|} \\ &= \left\langle \frac{h}{\|h\|}, a \right\rangle \left\langle \frac{h}{\|h\|}, \frac{h}{\|h\|} \right\rangle \frac{h}{\|h\|} \\ &= \left(\frac{h}{\|h\|} \otimes \frac{h}{\|h\|} \right)(a) \end{aligned}$$

hence $\left(\frac{h}{\|h\|} \otimes \frac{h}{\|h\|} \right)^2 = \frac{h}{\|h\|} \otimes \frac{h}{\|h\|}$ and $|g \otimes h| = \|g\|\,\|h\| \left(\frac{h}{\|h\|} \otimes \frac{h}{\|h\|} \right)$.

$$\begin{aligned} tr(|g \otimes h|) &= \sum_i \left\langle \|g\|\,\|h\| \left(\frac{h}{\|h\|} \otimes \frac{h}{\|h\|} \right) e_i, e_i \right\rangle = \|g\|\,\|h\| \sum_i \left\langle \left\langle \frac{h}{\|h\|}, e_i \right\rangle \frac{h}{\|h\|}, e_i \right\rangle \\ &= \|g\|\,\|h\| \sum_i \left\langle \frac{h}{\|h\|}, e_i \right\rangle \left\langle \frac{h}{\|h\|}, e_i \right\rangle = \|g\|\,\|h\| \left\langle \frac{h}{\|h\|}, \frac{h}{\|h\|} \right\rangle = \|g\|\,\|h\| \end{aligned}$$

Therefore $|B(g,h)| \le \|f\|\,\|g\|\,\|h\|$ and B is bounded with $\|B\| \le \|f\|$. Hence there exists a unique operator $S \in B(H)$ such that $\|S\| \le \|B\| \le \|f\|$ and $f(g \otimes h) = B(g,h) = \langle Sg, h\rangle$. We need to show that $f = L_S$.

Consider a self-adjoint operator $T \in S_1(H)$. Since T is also compact, it has a diagonal form $T = \sum_j \lambda_j e_j \otimes e_j$, where $\{e_j\}$ is ONB of H, $\{\lambda_j\}$ are real eigenvalues of T.

$$\begin{aligned} T^*T(a) &= \left(\sum_j \lambda_j e_j \otimes e_j\right)\left(\sum_j \lambda_j e_j \otimes e_j\right)(a) = \left(\sum_j \lambda_j e_j \otimes e_j\right)\sum_k \lambda_k \langle e_k, a\rangle e_k \\ &= \sum_k \lambda_k \langle e_k, a\rangle \sum_j \lambda_j \langle e_j, e_k\rangle e_j = \sum_k \lambda_k \langle e_k, a\rangle \lambda_k e_k = \sum_k \lambda_k^2 \langle e_k, a\rangle e_k \\ &= \sum_k \lambda_k^2 e_k \otimes e_k(a) \end{aligned}$$

So we are guessing that $(T^*T)^{1/2} = \sum_k |\lambda_k|\, e_k \otimes e_k$. Indeed,

$$\begin{aligned} (T^*T)^{1/2}(T^*T)^{1/2}(a) &= \sum_k |\lambda_k|\, e_k \otimes e_k \sum_j |\lambda_j|\, e_j \otimes e_j(a) \\ &= \sum_k |\lambda_k|\, e_k \otimes e_k \sum_j |\lambda_j| \langle e_j, a\rangle e_j \\ &= \sum_k |\lambda_k| \sum_j |\lambda_j| \langle e_j, a\rangle \langle e_k, e_j\rangle e_k \\ &= \sum_k |\lambda_k|^2 e_k \otimes e_k(a) \end{aligned}$$

Hence

$$\begin{aligned} \|T\|_1 = tr(|T|) = tr(\sum_k |\lambda_k|\, e_k \otimes e_k) &= \sum_i \left\langle \sum_k |\lambda_k|\, e_k \otimes e_k(e_i), e_i \right\rangle \\ = \sum_i \left\langle \sum_k |\lambda_k| \langle e_k, e_i\rangle e_k, e_i \right\rangle &= \sum_i \langle |\lambda_i|\, e_i, e_i\rangle = \sum_i |\lambda_i| \end{aligned}$$

Then

$$\begin{aligned} f(T) = f(\sum_k \lambda_k e_k \otimes e_k) &= \sum_k \lambda_k f(e_k \otimes e_k) = \sum_k \lambda_k \langle Se_k, e_k\rangle \\ &= \sum_k \langle S\lambda_k e_k, e_k\rangle = \sum_k \langle STe_k, e_k\rangle = tr(ST) \end{aligned}$$

We get $f(T) = tr(ST)$ for T - s.a. If T is an arbitrary operator in $S_1(H)$, then $T = T^+ - T^-$, where $T^+ = \frac{|T|+T}{2}$ and $T^- = \frac{|T|-T}{2}$. Since $S_1(H)$ is self-adjoint, T^+ and T^- are also in $S_1(H)$. Hence:

$$f(T) = f(T^+) - f(T^-) = tr(ST^+) - tr(ST^-) = tr(S(T^+ - T^-)) = tr(ST)$$

and $f = L_S$. Also from above: $\|L_S\| \le \|S\|$ and $\|L_S\| = \|f\| \ge \|S\|$, so $\|L_S\| = \|S\|$, and $L_S \leftrightarrow S$ is an isometric bijection between $(S_1(H))^*$ and $B(H)$. □

Consider the rank-one operator $g \otimes h$ for $g, h \in H$:

$$g \otimes h(a) = \langle h, a\rangle g$$

We have shown that $(g \otimes h)^* = h \otimes g$. Observe that if $\alpha \in \mathbb{C}$, $a, g, h \in H$, then:
$(\alpha g) \otimes h(a) = \langle h, a \rangle (\alpha g) = \alpha \langle h, a \rangle g = \alpha (g \otimes h)(a)$
$g \otimes (\alpha h)(a) = \langle \alpha h, a \rangle g = \bar{\alpha} \langle h, a \rangle g = \bar{\alpha}(g \otimes h)(a)$
So to make both expressions equal we introduce the space $\bar{H}$ with multiplication $\alpha.h := \bar{\alpha} h$. Then for $g, a \in H$, $h \in \bar{H}$:
$g \otimes (\alpha.h)(a) = \langle \bar{\alpha} h, a \rangle g = \alpha(g \otimes h)(a)$
We denote this by $g \otimes h \in H \otimes \bar{H}$. For any $g_1, g_2 \in H$, any $h_1, h_2 \in \bar{H}$, any $r \in \mathbb{C}$ hold:

(i) $(g_1 + g_2) \otimes h = g_1 \otimes h + g_2 \otimes h$
(ii) $g \otimes (h_1 + h_2) = g \otimes h_1 + g \otimes h_2$
(iii) $(rg) \otimes h = g \otimes (r.h) = r(g \otimes h)$

Elements in $H \otimes \bar{H}$ are of the form $\sum_{i=1}^{n} a_i \otimes b_i$ with $a_i \in H$ and $b_i \in \bar{H}$. We know that

$$\|g \otimes h\| = \sup_{\|a\| \le 1} \|g \otimes h(a)\| = \sup_{\|a\| \le 1} \|\langle a, h \rangle g\| = \sup_{\|a\| \le 1} |\langle a, h \rangle| \, \|g\| = \|h\| \, \|g\|$$

We want to have a norm on $H \otimes \bar{H}$ with the above property. Define:

$$\|u\|_\pi = \inf\{\sum_{i=1}^{n} \|x_i\| \, \|y_i\| \mid x_i \in H, y_i \in \bar{H}, u = \sum_{i=1}^{n} x_i \otimes y_i\}$$

$H \otimes \bar{H}$ with this norm denote by $H \otimes_\pi \bar{H}$.

Proposition 25. $S_1(H) \cong H \otimes_\pi \bar{H}$

Proof. If $T \in S_1(H)$, we need to show that T is in the closure of finite tensor operators. Since $S_1(H) \subset K(H)$ we have:
$|T| = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$, $\lambda_n \ge 0$, $\lambda_n \to 0$, $\{e_n\}$ is ONB of H and $tr(|T|) = \sum_{n=1}^{\infty} \lambda_n < \infty$

Define $T_k = \sum_{n=1}^{k} \lambda_n e_n \otimes e_n$. Then:

$$\||T| - T_k\|_1 = \left\| \sum_{n>k} \lambda_n e_n \otimes e_n \right\|_1 = tr(\sum_{n>k} \lambda_n e_n \otimes e_n) = \sum_i \left\langle \sum_{n>k} \lambda_n e_n \otimes e_n(e_i), e_i \right\rangle$$
$$= \sum_i \left\langle \sum_{n>k} \lambda_n \delta i, n e_n, e_i \right\rangle = \sum_{n>k} \langle \lambda_n e_n, e_n \rangle = \sum_{n>k} \lambda_n \xrightarrow[k\to\infty]{} 0$$

so $|T|$ is a limit of finite tensors, $T = U\,|T|$, therefore T is also in $H \otimes_\pi \bar{H}$. Hence $\|\cdot\|_\pi \le \|\cdot\|_1$.
On the other hand:

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_1 = \sup_{\|S\| \le 1} \left| tr\left(S \left(\sum_{i=1}^{n} x_i \otimes y_i \right) \right) \right| \le \sup_{\|S\| \le 1} \sum_{i=1}^{n} |tr(Sx_i \otimes y_i)|$$
$$\le \sup_{\|S\| \le 1} \sum_{i=1}^{n} \|Sx_i\| \, \|y_i\| \le \sum_{i=1}^{n} \|x_i\| \, \|y_i\|$$

Take infimum on both sides to get $\|\cdot\|_1 \le \|\cdot\|_\pi$. $\square$

Then we have:

$$S_1(H) \to H \otimes_\pi \bar{H} \to L_1(\nu)$$

where $g \otimes h \mapsto g \otimes h \mapsto f_{g,h}$, such that for any $a \in L^\infty(\nu)$ and for $\tilde{\pi} : L^\infty(\nu) \to B(H)$ holds $\langle g, \tilde{\pi}(a)h \rangle = \int a f_{g,h} d\nu$.

Denote by ψ the composition of the above maps, i.e. $\psi : S_1(H) \to L_1(\nu)$. It is well-defined, since $f_{g,h}$ is unique as a Radon-Nikodyn derivative and we have:
$f_{g_1+g_2,h} = f_{g_1,h} + f_{g_2,h}$
$f_{g,h_1+h_2} = f_{g,h_1} + f_{g,h_1}$
$f_{\alpha g,h} = \alpha f_{g,h} = f_{g,\alpha . h}$

The adjoint map $\psi^* : L_1(\nu)^* \to (S_1(H))^*$ is actually $\psi^* : L^\infty(\nu) \to B(H)$ and $\int a f_{g,h}\, d\nu = \langle g, \tilde{\pi}(a)h \rangle$, so we have the assignement $a \mapsto \tilde{\pi}(a) \in B(H)$.

Claim. $\psi^* = \tilde{\pi}$

Proof.

$$\begin{aligned}
\langle g, \tilde{\pi}(a)h \rangle &= \int a f_{g,h}\, d\nu = \langle a, f_{g,h}\nu \rangle = \langle a, \psi(g \otimes h) \rangle = \langle \psi^*(a), g \otimes h \rangle \\
&= tr((h \otimes g)\psi^*(a)) = tr(\psi^*(a)(h \otimes g)) = tr((\psi^*(a)h) \otimes g) \\
&= \sum_i \langle (\psi^*(a)h) \otimes g(e_i), e_i \rangle = \sum_i \langle \langle g, e_i \rangle (\psi^*(a)h), e_i \rangle = \langle g, \psi^*(a)h \rangle
\end{aligned}$$

□

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