

1. Problem 1: (10 points)

Find the following antiderivatives:

(a) (1 points)

$$\int \sin x dx = -\cos x + C$$

(b) (2 points)

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

(c) (3 points)

$$\int \frac{dx}{x^{17}} = \int x^{-17} dx = \frac{x^{-16}}{-16} + C = -\frac{1}{16x^{16}} + C$$

(d) (4 points)

$$\int (3 \sec^2 x + \frac{4}{x}) dx = 3 \int \sec^2 x dx + 4 \int \frac{dx}{x} = 3 \tan x + 4 \ln|x| + C$$

2. Problem 2: (10 points)

(a) (6 points) Approximate the area between $y = x^2$ and the x -axis for $0 \leq x \leq 2$ using n rectangles and a right endpoint calculation. Express your answer as a function of n , evaluating any summations that arise.

(b) (4 points) What is the exact area of the region described in part a)? Compute its value in two different ways.

a) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ ^{1 pt}, $x_0 = 0$, so $x_i = \frac{2i}{n}$. ^{1 pt}
 Then approx area = $\sum_{i=1}^n (\frac{2i}{n})^2 \cdot \frac{2}{n} = \frac{8}{n^3} \sum_{i=1}^n i^2 = \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}$ ^{2 pt}
 $= \frac{4}{3} \frac{(n+1)(2n+1)}{n^2}$ ^{2 pt}

b) Method 1: Area = $\lim_{n \rightarrow \infty} \frac{4}{3} \frac{(n+1)(2n+1)}{n^2} = \lim_{n \rightarrow \infty} \frac{4}{3} (1 + \frac{1}{n})(2 + \frac{1}{n}) = \frac{8}{3}$ ^{2 pt}

Method 2: Area = $\int_0^2 x^2 dx = \frac{x^3}{3} \Big|_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$ ^{2 pt}

Jinhyun

3. **Problem 3: (10 points)** In each of the following problems, evaluate the integral using a substitution. Indicate clearly the substitution you are using, and supply enough intermediate steps to convince me that you understand the substitution procedure.

(a) (5 points)

$$\int_0^{\pi/2} e^{\sin x} \cos x \, dx$$

$$= \int_0^1 e^u \, du$$

$$= e^u \Big|_0^1 = e - 1$$

$u = \sin x, \, du = \cos x \, dx \} +2$
 $x=0 \Rightarrow u = \sin 0 = 0 \} +1$
 $x = \pi/2 \Rightarrow u = \sin \frac{\pi}{2} = 1 \} +2$

(b) (5 points)

Red

$$\int \frac{x}{\sqrt{x+3}} \, dx$$

$u = x+3 \Rightarrow du = dx$

$$\int \frac{u-3}{\sqrt{u}} \, du = \int u^{\frac{1}{2}} - 3u^{-\frac{1}{2}} \, du$$

$$= \frac{2}{3} u^{\frac{3}{2}} - 6u^{\frac{1}{2}} + C = \frac{2}{3}(x+3)^{\frac{3}{2}} - 6(x+3)^{\frac{1}{2}} + C$$

$$\int \frac{x}{\sqrt{x-1}} \, dx$$

$u = x-1 \text{ so } x = u+1$
 $du = dx$

$$= \int \frac{u+1}{\sqrt{u}} \, du$$

$$= \int (u^{\frac{1}{2}} + u^{-\frac{1}{2}}) \, du = \frac{2}{3} u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C$$

$$= \frac{2}{3} (x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$$

4. **Problem 4: (10 points)**

Compute the Simpson's Rule approximation using $n = 6$ divisions of the interval for the integral

$$\int_1^4 \frac{x^2}{1+x^2} \, dx$$

x_0	x_1	x_2	x_3	x_4	x_5	x_6
1	1.5	2	2.5	3	3.5	4

ie $n=6$

$\{ x_n, n=1, \dots, 6 \}$
 $\Rightarrow (+3)$

$$S_6 = \frac{4-1}{3(6)} [f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)]$$

$$= \frac{1}{6} [\dots]$$

Number of $f(x_i)$
(4)

5. Problem 5: (10 points)

- (a) (3 points) Give a precise statement of Part I of the Fundamental Theorem of Calculus.
- (b) (3 points) What is the slope of the tangent line to

$$y = f(x) = \int_0^x (2 + \tan^3 t) dt$$

at the point $x = 0, y = 0$?

- (c) (4 points) If $g(x)$ is the function

$$g(x) = \int_0^{4x^2+2} t \ln t dt$$

compute $g'(x)$.

a) ¹ If f is continuous on $[a, b]$ and F is any antiderivative of f , then ¹

$$\int_a^b f(x) dx = F(b) - F(a)$$

b) $f'(x) = 2 + \tan^3 x$, so the slope at $x=0$
 is $f'(0) = 2 + \tan^3 0 = 2$

c) $g'(x) = (4x^2+2) \ln(4x^2+2) \cdot (4x^2+2)'$
 $= 8x(4x^2+2) \ln(4x^2+2)$



6. Problem 6: (10 points)

A car is travelling from west to east along a straight road with velocity $v(t) = 9 - t^2$. If the car's starting position is its location at $t = 0$:

- (a) (3 points) How far is the car from its starting position at time $t = 3$?
- (b) (3 points) At what times, if any, does the car come to a stop?
- (c) (4 points) What is the total distance travelled by the car from $t = 0$ to $t = 4$?

4B#5

$v(t) = 9 - t^2$ $(0, 9)$
 $(3, 0)$ $(0, 5)$

3

a) displacement = $\int_0^3 v(t) dt = \int_0^3 (9 - t^2) dt$
 $= (9t - \frac{t^3}{3}) \Big|_0^3$
 $= 27 - \frac{27}{3} = 18$

-1 $x(t) = 9t - \frac{t^3}{3}$
 no integral / FTC / initial position
 $x(3) - x(0)$

$\frac{128}{3} \approx 42.666$

so the car is 18 units from its start point.

3

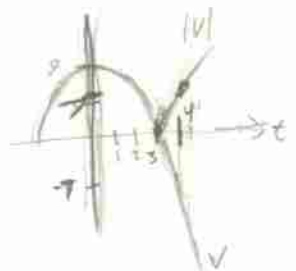
b) $v(t) = 9 - t^2 = 0 \Rightarrow 9 = t^2, t = \pm 3$.

-0 $t=3$ $t=4$

c) At $t = 3$ the car stops and changes direction.

Thus

Total distance = $\int_0^3 v(t) dt + \int_3^4 -v(t) dt$
 $= 18 + \int_3^4 (t^2 - 9) dt$
 $= 18 + (\frac{t^3}{3} - 9t) \Big|_3^4$
 $= 18 + \frac{64}{3} - 36 - (\frac{27}{3} - 27)$
 $= \frac{64}{3} \text{ units} \approx 21.33$



$2 \cdot \frac{128}{3} - \frac{115}{3} = \frac{141}{3}$
 $= 47$

-3 = ~~41~~

$\int_0^4 v dt = 9t - \frac{t^3}{3} \Big|_0^4 = 36 - \frac{64}{3} = \frac{44}{3} \approx 14.666$

$\frac{115}{3} \approx 38.333$

-2

$\int_0^3 v dt + \int_3^4 -v dt$

7. Problem 7: (10 points)

For each of the following questions, select exactly one answer. No justification is required.

(a) If A is the area under $y = x^2$ from $x = 0$ to $x = 1$ and L_{20} is the left endpoint approximation to A using 20 rectangles, then

- A. L_{20} over estimates A
 B. L_{20} under estimates A
 C. $L_{20} = A$
 Circle one: A B C



B

(b) Which of the following functions is an antiderivative of $2x + 7$?

- A. $x^2 + 7x - 3$
 B. $x^2 + 7x + 5$
 C. both
 D. neither
 Circle one: A B C D

← note: on Exam 4 B the answer is D.

2x+7

D C

(c) If f is the function $f(x) = \int_0^x (t^2 + 1)^{10} dt$, then

- A. f is decreasing everywhere
 B. f is increasing everywhere
 C. f is increasing on some intervals and decreasing on other intervals
 Circle one: A B C

$$f'(x) = (x^2 + 1)^{10} > 0$$

B

(d) The value of the limit

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{\frac{i}{n}} \frac{1}{n}$$

$$f(x) = \sqrt{x} \quad [0, 1]$$

$$\Delta x = \frac{1}{n} \quad x_i = \frac{i}{n}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{x_i} \Delta x = \int_0^1 \sqrt{x} dx = \left. \frac{2}{3} x^{3/2} \right|_0^1 = \frac{2}{3}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

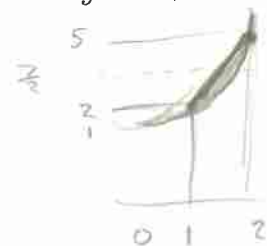
is

- A. $2/3$
 B. 0.6
 C. $1/2$
 Circle one: A B C

A

(e) The trapezoid approximation T_1 of the area under $y = 1 + x^2$ from $x = 1$ to $x = 2$ using one trapezoid is

- A. 3
 B. $7/2$
 C. neither
 Circle one: A B C



$$\frac{2+5}{2} (2-1)$$

B