1. Find all local extrema of the function

$$f(x) = \frac{x}{x^2+4}$$

The derivative of f is

$$\begin{aligned} f'(x) &= \frac{(x^2+4)(x)' - x(x^2+4)'}{(x^2+4)^2} \\ &= \frac{x^2+4-2x^2}{(x^2+4)^2} \\ &= \frac{4-x^2}{(x^2+4)^2} \end{aligned}$$

Solution:
The expression for f' shows that $f'(x)$ exists everywhere and so the critical numbers and hence the local extrema occur where $f'(x) = 0$. From the derivative this happens when $4 - x^2 = 0$, that is, at $x = -2$ and $x = 2$.
To determine which of these are local maxs or local mins, we need to do either of the following calculations:
First Derivative Test: From the expression for $f'(x)$ we see that $f'(x) > 0$ on $(-2, 2)$ and $f'(x) < 0$ on $(-\infty, -2)$ and $(2, \infty)$. This implies that f is increasing on $(-2, 2)$ and decreasing on $(-\infty, -2)$ and $(2, \infty)$. By the first derivative test, $x = -2$ is a local min and $x = 2$ is a local max.
Second Derivative Test: The second derivative of f is

$$\begin{aligned} f''(x) &= \frac{(x^2+4)^2(4-x^2)' - (4-x^2)((x^2+4)^2)'}{(x^2+4)^4} \\ &= \frac{(x^2+4)^2(-2x) - (4-x^2)2(x^2+4)(2x)}{(x^2+4)^4} \\ &= \frac{(x^2+4)(-2x) - (4-x^2)2(2x)}{(x^2+4)^3} \\ &= \frac{-2x^3 - 8x - 16x + 4x^3}{(x^2+4)^3} \\ &= \frac{2x^3 - 24x}{(x^2+4)^3} \\ &= \frac{2x(x^2-12)}{(x^2+4)^3} \end{aligned}$$

Thus $f''(2) = 2 \cdot 2 \cdot (-8)/(8)^3 < 0$ and so $x = 2$ is a local max. Similarly $f''(-2) = 2 \cdot (-2) \cdot (-8)/(8)^3 > 0$ and so $x = -2$ is a local min.

2. Determine the intervals where the graph of

$$g(x) = x^3 - 9x^2 + 2x - 1$$

is concave up and concave down. Express your answer in interval notation. Is there an inflection point?

Solution: The derivative of g is

$$g'(x) = 3x^2 - 18x + 2$$

Therefore the second derivative of g is

$$\begin{aligned} g''(x) &= 6x - 18 \\ &= 6(x - 3) \end{aligned}$$

We see that $g''(x) > 0$ on $(3, \infty)$ and $g''(x) < 0$ on $(-\infty, 3)$. This implies that g is concave up on $(3, \infty)$ and concave down on $(-\infty, 3)$. This also shows that $(3, g(3))$ is an inflection point.