Assignment 11 Solutions:

11.3.10 By Theorem 8

$$\int_1^\infty \frac{dx}{x\sqrt{x}} \le \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}} \le \frac{1}{1\sqrt{1}} + \int_1^\infty \frac{dx}{x\sqrt{x}}.$$

So we compute $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{t\to\infty} -\frac{2}{\sqrt{x}}\Big]_1^t = \lim_{t\to\infty}\left(-\frac{2}{\sqrt{t}}+2\right) = 2$

Thus $\displaystyle 2 \le \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}} \le 3.$

11.3.12 By Theorem 8

$$\int_1^\infty \frac{dx}{1+x^2} \le \sum_{k=1}^{\infty} \frac{1}{1+k^2} \le \frac{1}{1+1^2} + \int_1^\infty \frac{dx}{1+x^2}$$

This is OK because $\frac{1}{1+x^2}$ is decreasing on $(1,\infty)$. And

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \arctan x\Big]_1^t = \lim_{t\to\infty}(\arctan t - \arctan 1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

To handle the sum from $k=0$ to ∞, add in the term for $k=0$:

$$1 + \frac{\pi}{4} \le \sum_{k=0}^{\infty} \frac{1}{1+k^2} \le 1 + \frac{1}{2} + \frac{\pi}{4}$$

11.3.18 Since $\displaystyle \frac{1}{2e^j} \le \frac{1}{j+e^j} \le \frac{1}{e^j} = \left(\frac{1}{e}\right)^j$

and $\displaystyle\sum_{j=0}^{\infty}\left(\frac{1}{e}\right)^j$ is a geometric series with $r = \frac{1}{e}$, the original series converges. Moreover, this gives us

$$\frac{\frac{1}{2}}{1-\frac{1}{e}} = \sum_{j=0}^{\infty} \frac{1}{2}\left(\frac{1}{e}\right)^j \le \sum_{j=0}^{\infty} \frac{1}{j+e^j} \le \sum_{j=0}^{\infty}\left(\frac{1}{e}\right)^j = \frac{1}{1-1/e}$$

ie $\displaystyle \frac{e}{2(e-1)} \le \sum_{j=0}^{\infty} \frac{1}{j+e^j} \le \frac{e}{e-1}$