Assignment 12 Solutions

11.6.20 $f(x) = \ln(1+x^2)$.

Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}(-1)^{k+1}$ from Example 5, replace x by x^2 to get

$$f(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k}}{k}$$

and it converges for $|x^2| < 1$, i.e. $|x| < 1$

11.6.22 $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$.

From the expansion in 11.6.20 we know that

$$\ln(1-x) = (-x) - \left(\frac{(-x)^2}{2}\right) + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \cdots$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

Thus, $f(x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)$

$$= 2x + \frac{2x^3}{3} + 2\frac{x^5}{5} + \cdots$$

11.6.32 $L = \lim_{x\to 0} \frac{\arctan x}{x}$

By L'Hopital: both top and bottom tend to 0, so

$$L = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1.$$

By series: From pg 593 we know

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

So $\lim_{x\to 0} \frac{\arctan x}{x} = \lim_{x\to 0} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots\right) = 1$.

11.7.4 $f(x) = \dfrac{x}{1-x^3}$.

a) Begin with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

and so $$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots$$

and thus

$$f(x) = x + x^4 + x^7 + x^{10} + \cdots = \sum_{k=0}^{\infty} x^{3k+1}$$

b) By the ratio test,

$$L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{|x|^{3(k+1)+1}}{|x|^{3k+1}}$$

$$= \lim_{k\to\infty} |x|^3$$

So we need $|x|^3 < 1$, ie $|x| < 1$. The radius of convergence is 1.

c) $f'(x) = 1 + 4x^3 + 7x^6 + 10x^9 + \cdots$

$f''(x) = 12x^2 + 42x^5 + 90x^8 + \cdots$

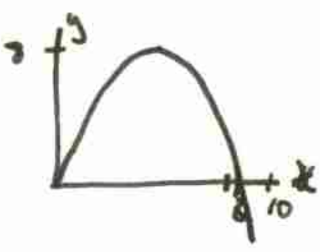
d) $\displaystyle\int_0^x f(t)\,dt = \frac{x^2}{2} + \frac{x^5}{5} + \frac{x^8}{8} + \frac{x^{11}}{11} + \cdots$

V.1.26 $p(0) = (0,0)$, $v(0) = (4,4)$, $a(t) = (0,-1)$

a) $v(t) = \displaystyle\int_0^t a(s)\,ds = (4, -t+4)$

$p(t) = \displaystyle\int_0^t v(s)\,ds = \left(4s\Big]_0^t, -\frac{s^2}{2} + 4s\Big]_0^t \right)$

$$= \left(4t, -\frac{t^2}{2} + 4t\right)$$

b) 

Looks like a parabola

c) $x = 4t$, $y = -\frac{t^2}{2} + 4t$, so $t = \frac{x}{4}$, ie $y = -\frac{1}{2}\left(\frac{x}{4}\right)^2 + x = -\frac{x^2}{32} + x$, a parabola.

d) $\text{arclength} = \displaystyle\int_0^{10} \sqrt{(x')^2 + (y')^2}\,dt = \int_0^{10} \sqrt{4^2 + (-t+4)^2}\,dt$

Set $-t+4 = 4u \Rightarrow -dt = 4du$, $t=0 \Rightarrow u = 1$

$t = 10 \Rightarrow u = -\frac{3}{2}$

$$\Rightarrow \text{arclength} = \int_1^{-3/2} \sqrt{4^2+4^2u^2}\,(-4\,du)$$
$$= 16 \int_{-3/2}^{1} \sqrt{1+u^2}\,du = 49.5561$$

V.1.30 $x(t) = t^3-3t$, $y(t) = t^2-2t$

a) $x' = 3t^2-3$, $y' = 2t-2$.

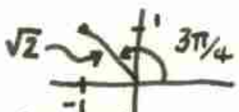
The particle comes to a stop when its velocity is $(0,0)$, i.e. $(3t^2-3, 2t-2) = (0,0)$, so $t=1$. (why?)

Where?: $x(1) = 1-3 = -2$, $y(1) = 1-2 = -1 \Rightarrow$ at $(-2,-1)$.

b) Straight up or straight down occurs when $x'=0, y'\neq 0$.
So $x'=0 \Leftrightarrow 3t^2-3=0$ i.e. $t=\pm 1$. Since the particle is at a stop when $t=1$, we consider only $t=-1$. At this point we are at $p(-1) = ((-1)^3-3(-1), (-1)^2-2(-1)) = (-1+3, 1+2) = (2,3)$.

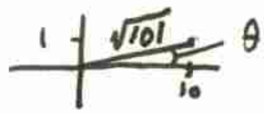
c) This is the analogous case when $y'=0, x'\neq 0$. But this never occurs! (why?)

V.2.4 Cartesian Point $(-1,1)$



Polar Points: $(\sqrt{2}, \frac{3\pi}{4})$, $(-\sqrt{2}, \frac{7\pi}{4})$, $(\sqrt{2}, \frac{11\pi}{4})$

V.2.8 Cartesian Point $(10,1)$



Polar Points: $(\sqrt{101}, 0.0997)$
$(-\sqrt{101}, 0.0997+\pi)$
$(\sqrt{101}, 0.0997+2\pi)$

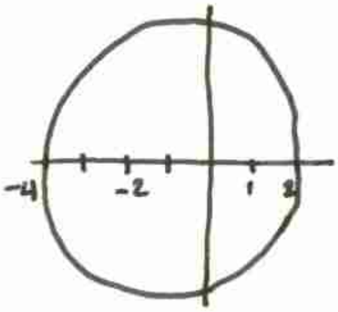
$\tan\theta = \frac{1}{10} \Rightarrow \theta = 0.0997$

V.2.18 $r=4 \Leftrightarrow \sqrt{x^2+y^2}=4 \Leftrightarrow x^2+y^2=4^2$ a circle at $(0,0)$ of radius 4.

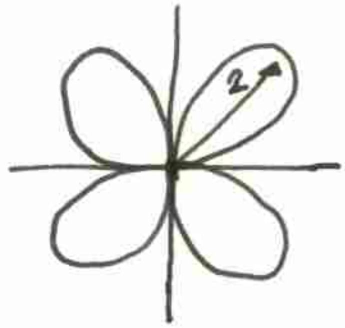
V.2.20 $r=2\sin\theta \Leftrightarrow r^2 = 2r\sin\theta \Leftrightarrow x^2+y^2=2y$
$\Leftrightarrow x^2+y^2-2y+1 = 1 \Leftrightarrow x^2+(y-1)^2 = 1^2$
circle at $(0,1)$ of radius 1.

V.2.22 $\quad y=4 \iff r\cos\theta=4 \iff r=\frac{4}{\cos\theta}=4\sec\theta.$

V.2.30 $\quad r=3-\cos\theta$

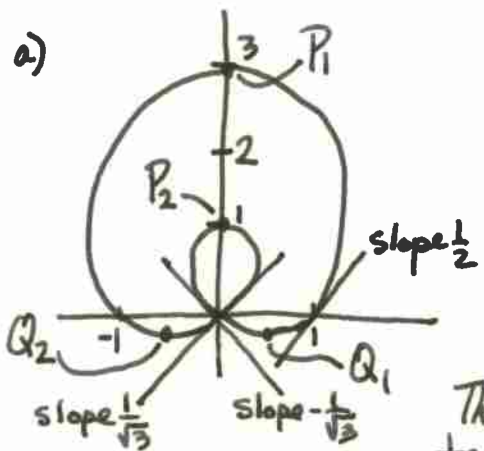


V.2.36 $\quad r=2\sin(2\theta)$



V.3.4 $\quad r=1+2\sin\theta$

a)



$$\frac{dy}{dx}=\frac{2\cos\theta\sin\theta+(1+2\sin\theta)\cos\theta}{2\cos\theta\cos\theta-(1+2\sin\theta)\sin\theta}$$

$$=\frac{\cos\theta+4\sin\theta\cos\theta}{2\cos^2\theta-2\sin^2\theta-\sin\theta}$$

b) at polar point $(1,0)$ we have $\theta=0$, so

$$\frac{dy}{dx}=\frac{1+0}{2+0}=\frac{1}{2}$$

The cartesian point is also $(1,0)$ (why?), so the tangent line is $\quad y=\frac{1}{2}(x-1)$.

c) At $(0,\frac{7\pi}{6})$ we have $\theta=\frac{7\pi}{6}$. Since $\cos\frac{7\pi}{6}=-\frac{\sqrt{3}}{2}$, $\sin\frac{7\pi}{6}=-\frac{1}{2}$

Thus $\quad \frac{dy}{dx}=\frac{-\frac{\sqrt{3}}{2}+4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{2\left(\frac{\sqrt{3}}{2}\right)^2-2\left(\frac{1}{2}\right)^2-\left(-\frac{1}{2}\right)}=\frac{1}{\sqrt{3}}$

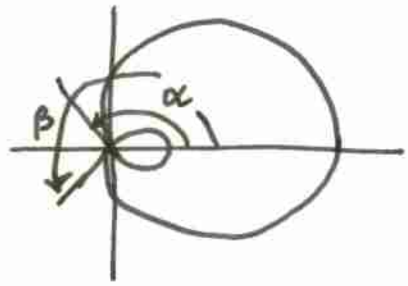
d) At $(0,\frac{11\pi}{6})$ we have $\theta=\frac{11\pi}{6}$. A similar calculation, or by symmetry, gives $\frac{dy}{dx}=-\frac{1}{\sqrt{3}}$.

e) Tangents are horizontal when $\frac{dy}{dx}=0$, ie $\cos\theta(1+4\sin\theta)=0$.

$\cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}, \frac{3\pi}{2}$. These are the points P_1 and P_2.

$1+4\sin\theta=0 \Rightarrow \sin\theta=-\frac{1}{4} \Rightarrow \theta=-.25268,\ +.25268+\pi$. These are the points Q_1 and Q_2

V.3.10



We need the angles α and β. These are angles at which $r = 1 + 2\cos\theta$ is zero, ie

$1 + 2\cos\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\Rightarrow \alpha = \frac{2\pi}{3}, \beta = \frac{4\pi}{3}$

So our area is $A = \frac{1}{2}\int_{\frac{2\pi}{3}}^{4\pi/3} (1 + 2\cos\theta)^2 \, d\theta$

$$= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 4\cos\theta + 2\cos 2\theta + 2)\, d\theta$$

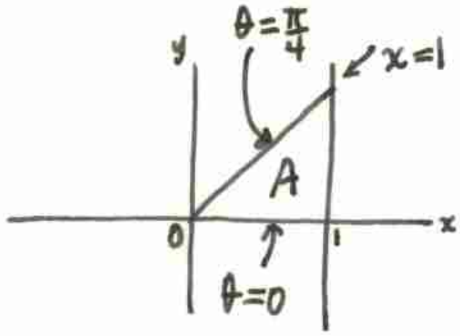
$$= \frac{1}{2}\left(3\theta + 4\sin\theta + \sin 2\theta\right)\Big]_{2\pi/3}^{4\pi/3}$$

$$= \pi - \frac{3\sqrt{3}}{2}$$

V.3. $\underbrace{r = \sec\theta}$, $\theta = 0$, $\theta = \pi/4$

$r = \frac{1}{\cos\theta} \Leftrightarrow \quad r\cos\theta = 1$

$\Leftrightarrow \quad x = 1$



A = area of the triangle

$= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$.

Equivalently, $A = \frac{1}{2}\int_0^{\pi/4} \sec^2\theta \, d\theta = \frac{1}{2}\tan\theta\Big]_0^{\pi/4} = \frac{1}{2}\left(\tan\frac{\pi}{4} - \tan 0\right) = \frac{1}{2}$