

## Assignment 9 Solutions

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$$9.1.20 \quad f(x) = \frac{1}{x^2}, \quad x_0 = 1 \Rightarrow f(1) = 1$$

$$f'(x) = -\frac{2}{x^3}, \quad f'(1) = -2!$$

$$f''(x) = \frac{3!}{x^4}, \quad f''(1) = 3!$$

$$f'''(x) = -\frac{4!}{x^5}, \quad f'''(1) = -4!$$

$$f^{(4)}(x) = \frac{5!}{x^6}, \quad f^{(4)}(1) = 5!$$

$$f^{(5)}(x) = -\frac{6!}{x^7}, \quad f^{(5)}(1) = -6!$$

$$f^{(6)}(x) = \frac{7!}{x^8}$$

$$\begin{aligned} \text{So } P_5(x) &= 1 - 2!(x-1) + \frac{3!}{2!}(x-1)^2 - \frac{4!}{3!}(x-1)^3 + \frac{5!}{4!}(x-1)^4 - \frac{6!}{5!}(x-1)^5 \\ &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - 6(x-1)^5 \end{aligned}$$

A plot shows that  $|f(x) - P_5(x)| \leq .01$  for  $.69 \leq x \leq 1.36$

$$9.1.22 \quad f(x) = \sqrt{x}, \quad x_0 = 4 \Rightarrow f(4) = 2$$

$$f'(x) = \frac{1}{2} x^{-1/2}, \quad f'(4) = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4} x^{-3/2}, \quad f''(4) = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8} x^{-5/2}, \quad f'''(4) = \frac{3}{8 \times 2^5}$$

$$f^{(4)}(x) = -\frac{15}{16} x^{-7/2}, \quad f^{(4)}(4) = -\frac{15}{16 \times 2^7}$$

$$f^{(5)}(x) = \frac{105}{32} x^{-9/2}, \quad f^{(5)}(4) = \frac{105}{32 \times 2^8}$$

$$\begin{aligned} \text{So } P_5(x) &= 2 + \frac{1}{4}(x-4) - \frac{1}{32 \times 2!}(x-4)^2 + \frac{3}{8 \times 2^5 \times 3!}(x-4)^3 \\ &\quad - \frac{15}{16 \times 2^7 \times 4!}(x-4)^4 + \frac{105}{32 \times 2^8 \times 5!}(x-4)^5 \\ &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{5}{16384}(x-4)^4 \\ &\quad + \frac{7}{131072}(x-4)^5 \end{aligned}$$

A plot shows that  $|f(x) - P_5(x)| \leq .01$  when  $1.2 \leq x \leq 7.5$

$$9.2.6 \quad f(x) = \ln x, \quad x_0 = 1 \Rightarrow f(1) = \ln 1 = 0$$

$$f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3}, \quad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{3!}{x^4}, \quad f^{(4)}(1) = -3!$$

$$f^{(5)}(x) = \frac{4!}{x^5}$$

$$\begin{aligned} \text{Thus } P_4(x) &= 0 + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \end{aligned}$$

a) On the interval  $[\frac{1}{2}, \frac{3}{2}]$ ,  $|f^{(5)}(x)| \leq \frac{4!}{(\frac{1}{2})^5} = 4! \cdot 2^5 = K_5$   
and so Thm 2 says that

$$|f(x) - P_4(x)| \leq \frac{K_5}{5!} |x-1|^5 \leq \frac{4! \cdot 2^5}{5!} \left(\frac{1}{2}\right)^5 = \frac{1}{5} = .2 \text{ on } \left[\frac{1}{2}, \frac{3}{2}\right].$$

b) A plot shows that the actual error is approximately .01.

$$9.2.10 \quad f(x) = e^{-x^2}, \quad x_0 = 0 \Rightarrow f(0) = 1$$

$$f'(x) = -2x e^{-x^2}, \quad f'(0) = 0$$

$$\begin{aligned} f''(x) &= -2e^{-x^2} - 2x(-2xe^{-x^2}), \quad f''(0) = -2 \\ &= -2e^{-x^2} + 4x^2 e^{-x^2} \end{aligned}$$

$$\begin{aligned} f'''(x) &= -2(-2xe^{-x^2}) + 8xe^{-x^2} + 4x^2(-2xe^{-x^2}) \\ &= (4x + 8x - 8x^3) e^{-x^2} \\ &= 4x(3 - 2x^2) e^{-x^2} \end{aligned}$$

We want to approximate  $\int_0^{\frac{2}{3}} f(x) dx$ , so apply Thm 2 with the interval  $[-\frac{2}{3}, \frac{2}{3}]$ . A plot of  $f'''$  shows that

$$|f'''(x)| \leq 4 \text{ on } \left[-\frac{2}{3}, \frac{2}{3}\right]. \text{ Thus take } K_3 = 4, \text{ and then}$$

$$\text{Thm 2 says } |f(x) - P_2(x)| \leq \frac{4}{3!} |x|^3 \text{ on } \left[-\frac{2}{3}, \frac{2}{3}\right].$$

This implies that

$$\begin{aligned}
 & \left| \int_0^{2/3} f(x) dx - \int_0^{2/3} P_2(x) dx \right| \\
 &= \left| \int_0^{2/3} (f(x) - P_2(x)) dx \right| \\
 &\leq \int_0^{2/3} |f(x) - P_2(x)| dx \\
 &\leq \int_0^{2/3} \frac{4}{3!} x^3 dx \\
 &= \frac{4}{3!} \left[ \frac{x^4}{4} \right]_0^{2/3} = \frac{4}{3!} \cdot \frac{1}{4} \left( \frac{2}{3} \right)^4 = .033
 \end{aligned}$$

9.2.14  $L = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}.$

To estimate  $L$ , let us use  $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$  in place of  $e^x$ :

$$L = \lim_{x \rightarrow 0} \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} - 1 - x}{x^2} = \lim_{x \rightarrow 0} \left( \frac{1}{2} + \frac{x}{6} \right) = \frac{1}{2}.$$

To prove that this is the correct limit, we need to show that

$$\left| \frac{e^x - 1 - x}{x^2} - \frac{1}{2} \right|$$

is very small for  $x$  near 0. Note that

$$\frac{e^x - 1 - x}{x^2} - \frac{1}{2} = \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^2} = \frac{e^x - P_2(x)}{x^2}.$$

Theorem 2 helps us estimate  $|e^x - P_2(x)|$ . Since we want to know what happens as  $x \rightarrow 0$ , choose any interval containing  $x=0$ , say  $[-1, 1]$ . With  $f(x) = e^x$  we know  $f'''(x) = e^x$  and so  $|f'''(x)| = e^x \leq e = K_3$  for  $-1 \leq x \leq 1$ . Theorem 2 then says

$$|e^x - P_2(x)| \leq \frac{K_3}{3!} |x|^3 = \frac{e}{3!} |x|^3$$

Thus,  $\left| \frac{e^x - 1 - x}{x^2} - \frac{1}{2} \right| \leq \frac{1}{x^2} \frac{e}{3!} |x|^3 = \frac{e}{3!} |x|$ . This last term goes to 0 as  $|x| \rightarrow 0$ , so the limit is  $\frac{1}{2}$ .

9.2.16  $m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ . We need to approximate this near  $v=0$  by a quadratic function. Use

$$P_2(v) = m(0) + m'(0)v + \frac{m''(0)}{2!}v^2.$$

$$m(0) = m_0$$

$$m'(v) = -\frac{1}{2}m_0 \left(1 - \frac{v^2}{c^2}\right)^{-3/2} \left(-\frac{2v}{c^2}\right) \Rightarrow m'(0) = 0$$

$$m''(v) = -\frac{1}{2}m_0 \left(-\frac{3}{2}\right) \left(1 - \frac{v^2}{c^2}\right)^{-5/2} \left(-\frac{2v}{c^2}\right)^2 + m_0 \frac{1}{c^2} \left(1 - \frac{v^2}{c^2}\right)^{-3/2} \Rightarrow m''(0) = \frac{m_0}{c^2}.$$

Thus  $m(v) \approx m_0 + \frac{m_0}{2c^2}v^2$ .

9.2.18  $f(0) = 26, f'(0) = 22, f''(0) = -16, f'''(0) = 12,$   
 $|f^{(4)}(x)| \leq 7x^4$  for  $|x| \leq 3$ .

$$P_3(x) = 26 + 22x - 8x^2 + 2x^3.$$

We want to estimate  $f(1)$ , so apply Thm 2 on the interval  $[-1, 1]$ .

We see that  $|f^{(4)}(x)| \leq 7$  for  $|x| \leq 1$ , so take  $K_4$  to be 7.

$$\text{Thus } |f(x) - P_3(x)| \leq \frac{K_4}{4!}|x|^4 = \frac{7}{24} \text{ for } |x| \leq 1.$$

In particular

$$|f(1) - P_3(1)| \leq \frac{7}{24}.$$

$$\text{ie } -\frac{7}{24} \leq f(1) - P_3(1) \leq \frac{7}{24}$$

$$\text{ie } P_3(1) - \frac{7}{24} \leq f(1) \leq P_3(1) + \frac{7}{24}$$

Since  $P_3(1) = 26 + 22 - 8 + 2 = 42$ , we have

$$\underline{42 - \frac{7}{24}} \leq f(1) \leq \underline{42 + \frac{7}{24}}$$

lower bound

upper bound