Supplementary Notes & Problems on Center of Mass

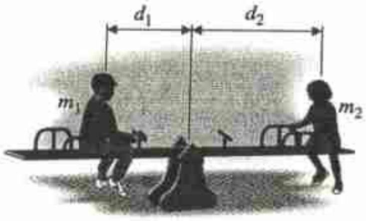

FIGURE 5.51a
Balancing two masses

The concept of the *first moment,* like work, involves force and distance. Moments are used to solve problems of balance and rotation. Consider two children on a playground seesaw (or teeter-totter). Suppose that the child on the left in Figure 5.51a is heavier (i.e., has larger mass) than the child on the right. If the children sit an equal distance from the pivot point, you know what will happen: the left side will be pulled down. However, the children can balance each other if the heavier child moves closer to the pivot point. That is, the balance is determined both by weight (force) and distance from the pivot point. If the children have masses m_1 and m_2 and are sitting at distances d_1 and d_2, respectively, from the pivot point, then they balance each other if and only if

$$m_1 d_1 = m_2 d_2. \tag{6.3}$$

Let's turn the problem around slightly. Suppose there are two objects, of mass m_1 and m_2, located at x_1 and x_2, respectively, with $x_1 < x_2$. We consider the objects to be **point-masses.** That is, each is treated as a single point, with all of the mass concentrated at that point (see Figure 5.51b).

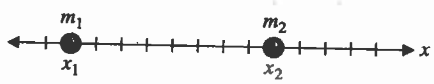

FIGURE 5.51b
Two point-masses

Suppose that you want to find the **center of mass** $\bar{x}$, that is, the location at which you could place the pivot of a seesaw and have the objects balance. From the balance equation (6.3), you'll need $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$. Solving this equation for $\bar{x}$ gives us

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Notice that the denominator in this equation is the total mass of the "system" (i.e., the total mass of the two objects). The numerator of this expression is called the **first moment** of the system.

More generally, for a system of n masses $m_1, m_2, \ldots, m_n$, located at $x = x_1, x_2, \ldots, x_n$, respectively, the center of mass $\bar{x}$ is given by the first moment divided by the total mass, that is,

Center of mass

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}.$$

Now, suppose that we wish to find the mass and center of mass of an object of variable density that extends from $x = a$ to $x = b$. Here, we assume that the density function $\rho(x)$ (measured in units of mass per unit length) is known. Note that if the density is a constant ρ, the mass of the object is simply given by $m = \rho L$, where $L = b - a$ is the length of the object. On the other hand, if the density varies throughout the object, we can approximate the mass by dividing the interval $[a, b]$ into n pieces of equal width $\Delta x = \frac{b-a}{n}$. On each subinterval $[x_{i-1}, x_i]$, the mass is approximately $\rho(c_i)\Delta x$, where c_i is a point in the subinterval. The total mass is then approximately

$$m \approx \sum_{i=1}^{n} \rho(c_i)\,\Delta x.$$

You should recognize this as a Riemann sum, which approaches the total mass as $n \to \infty$,

Mass

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(c_i)\,\Delta x = \int_a^b \rho(x)\,dx. \tag{6.4}$$

EXAMPLE 6.5 Computing the Mass of a Baseball Bat

A 30-inch baseball bat can be represented approximately by an object extending from $x = 0$ to $x = 30$ inches, with density $\rho(x) = \left(\frac{1}{46} + \frac{x}{690}\right)^2$ slugs per inch. The density takes into account the fact that a baseball bat is similar to an elongated cone. Find the mass of the object.

Solution From (6.4), the mass is given by

$$m = \int_0^{30} \left(\frac{1}{46} + \frac{x}{690}\right)^2 dx$$
$$= \frac{690}{3}\left(\frac{1}{46} + \frac{x}{690}\right)^3 \Big|_0^{30} = \frac{690}{3}\left[\left(\frac{1}{46} + \frac{30}{690}\right)^3 - \left(\frac{1}{46}\right)^3\right]$$
$$\approx 6.144 \times 10^{-2} \text{ slug.}$$

To compute the weight (in ounces), multiply the mass by $32 \cdot 16$. The bat weighs roughly 31.5 ounces. ■

To compute the first moment for an object of nonconstant density $\rho(x)$ extending from $x = a$ to $x = b$, we again divide the interval into n equal pieces. From our earlier argument, for each $i = 1, 2, \ldots, n$, the mass of the ith slice of the object is approximately $\rho(c_i)\,\Delta x$, for any choice of $c_i \in [x_{i-1}, x_i]$. We then represent the ith slice of the object with a particle of mass $m_i = \rho(c_i)\,\Delta x$ located at $x = c_i$. We can now think of the original object as having been approximated by n distinct point-masses, as indicated in Figure 5.52.

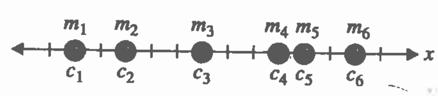

FIGURE 5.52
Six point-masses

Notice that the first moment M_n of this approximate system is

$$M_n = [\rho(c_1)\,\Delta x]c_1 + [\rho(c_2)\,\Delta x]c_2 + \cdots + [\rho(c_n)\,\Delta x]c_n$$
$$= [c_1\rho(c_1) + c_2\rho(c_2) + \cdots + c_n\rho(c_n)]\,\Delta x = \sum_{i=1}^{n} c_i\rho(c_i)\Delta x.$$

Taking the limit as $n \to \infty$, the sum approaches the first moment

First moment

$$M = \lim_{n\to\infty} \sum_{i=1}^{n} c_i\rho(c_i)\,\Delta x = \int_a^b x\rho(x)\,dx. \quad (6.5)$$

The center of mass of the object is then given by

Center of mass

$$\bar{x} = \frac{M}{m} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}. \quad (6.6)$$

EXAMPLE 6.6 **Finding the Center of Mass (Sweet Spot) of a Baseball Bat**

Find the center of mass of the baseball bat from example 6.5.

Solution From (6.5), the first moment is given by

$$M = \int_0^{30} x\left(\frac{1}{46} + \frac{x}{690}\right)^2 dx = \left[\frac{x^2}{4232} + \frac{x^3}{47{,}610} + \frac{x^4}{1{,}904{,}400}\right]_0^{30} \approx 1.205.$$

Recall that we had already found the mass to be $m \approx 6.144 \times 10^{-2}$ slug and so, from (6.6), the center of mass of the bat is

$$\bar{x} = \frac{M}{m} \approx \frac{1.205}{6.144 \times 10^{-2}} \approx 19.6 \text{ inches.}$$

Note that for a baseball bat, the center of mass is one candidate for the so-called "sweet spot" of the bat, the best place to hit the ball. ■

Center of Mass Problems

21. Compute the mass and center of mass of an object with density $\rho(x) = \frac{x}{6} + 2$ kg/m, $0 \le x \le 6$. Briefly explain in terms of the density function why the center of mass is not at $x = 3$.

22. Compute the mass and center of mass of an object with density $\rho(x) = 3 - \frac{x}{6}$ kg/m, $0 \le x \le 6$. Briefly explain in terms of the density function why the center of mass is not at $x = 3$.

23. Compute the weight in ounces of an object extending from $x = -3$ to $x = 27$ with density $\rho(x) = \left(\frac{1}{46} + \frac{x+3}{690}\right)^2$ slugs/in.

24. Compute the weight in ounces of an object extending from $x = 0$ to $x = 32$ with density $\rho(x) = \left(\frac{1}{46} + \frac{x+3}{690}\right)^2$ slugs/in.

25. Compute the center of mass of the object in exercise 23. This object models the baseball bat of example 6.5 "choked up" (held 3 inches up the handle). Compare the masses and centers of mass of the two bats.

26. Compute the center of mass of the object in exercise 24. This object models a baseball bat that is 2 inches longer than the bat of example 6.5. Compare the masses and centers of mass of the two bats.

27. Compute the mass and weight in ounces and center of mass of an object extending from $x = 0$ to $x = 30$ with density $\rho(x) = 0.00468\left(\frac{3}{16} + \frac{x}{60}\right)$ slugs/in.

28. The object in exercise 27 models an aluminum baseball bat (hollow and $\frac{1}{4}$ inch thick). Compare the mass and center of mass to the wooden bat of example 6.5. Baseball experts claim that it is easier to hit an inside pitch (small x value) with an aluminum bat. Explain why your calculations indicate that this is true.

29. The accompanying figure shows the outline of a model rocket. Assume that the vertical scale is 3 units high and the horizontal scale is 6 units wide. Use basic geometry to compute the area of each of the three regions of the rocket outline. Assuming a

Center of Mass Solutions

19. maximum thrust: $\frac{30}{e} \approx 11.036$; impulse: $90 - \frac{270}{e^2} \approx 53.459$

21. $m = 15$ kg, $\bar{x} = \frac{16}{5}$ m; heavier to right of center

23. 0.0614 slug, 31.5 oz

25. 16.6 in.; same mass, $\bar{x}$ differs by 3

27. 0.0614 slug, 31.4 oz; $\bar{x} = 17.86$ in.

29. 2, 4, $\frac{1}{2}$; $\frac{5}{12}$, 3, $\frac{16}{3}$

31. $\left(\frac{8}{3}, 2\right)$ **33.** (0, 1.6) **35.** 8,985,600 lb

37. 196,035 lb **39.** 12,252 lb **41.** $-16\sqrt{5}$ ft/s

43. 10,667 hp **45.** 27.22, 20.54, 24.53% **47.** $\frac{1}{4}\rho\pi a^3 b$

49. $\frac{\text{midsized}}{\text{wooden}} \approx 1.35$; $\frac{\text{oversized}}{\text{wooden}} \approx 1.78$