Black: Rigorous calculations　　Blue: Rough reasoning　　Red: My comments

Sample Test No. 2
Math 130 - Calculus and Analytical Geometry II

Value　Problem

1. Find the limit of the following sequences, justifying each of your steps:

10　a) $\{a_n\} = \left\{(3\ln n)^{1/n}\right\}$

Since $\ln n \to \infty$ and $1/n \to 0$, this is a ∞^0 form. Thus we apply logs and then use L'Hôpital's rule:

Introduce your starting point

Let $L = \lim_{n\to\infty} (3\ln n)^{1/n}$. Then (applying a contin function)

$$\ln L = \lim_{n\to\infty} \ln\left((3\ln n)^{1/n}\right)$$

Can pull lim outside a contin. fn.

$$= \lim_{n\to\infty} \frac{1}{n}\ln(3\ln n)$$

$$= \lim_{n\to\infty} \frac{\ln 3 + \ln(\ln n)}{n} \leftarrow \frac{\infty}{\infty} \text{ form}$$

new identif'n of indeterminate form

L'Hopital

$$= \lim_{n\to\infty} \frac{\frac{1}{\ln n}\cdot\frac{1}{n}}{1}$$

$$= 0$$

Thus $L = e^{\ln L} = e^0 = 1.$

undo transformation

10　b) $\{a_n\} = \left\{\dfrac{n\sin n}{\sqrt{n^3+1}}\right\}$

Since $\sin n$ alternates in the interval $-1 \le \sin n \le 1$, let's use the squeeze rule here.

excellent thought process, but needs precision

Note first that $\dfrac{n}{\sqrt{n^3+1}} \le \dfrac{n}{\sqrt{n^3}} = \dfrac{1}{\sqrt{n}} \to 0$ as $n\to\infty$

smaller denom

justification of inequality

Therefore,

$$-\frac{n}{\sqrt{n^3+1}} \le \frac{n\sin n}{\sqrt{n^3+1}} \le \frac{n}{\sqrt{n^3+1}}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 \qquad\qquad\qquad 0$$

explicitly demonstrate squeeze

We conclude that $\lim_{n\to\infty} \dfrac{n\sin n}{\sqrt{n^3+1}} = 0.$

10 2. Does the following series converge or diverge, and why?

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Worth a little since it tells me you are thinking correctly

Since the general term $a_n = \frac{1}{n(\ln n)^2}$ contains logs, and the derivative of $\ln n$, ie $\frac{1}{n}$, also appears, this is a good place to use the integral test.

Consider $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{h\to\infty} \int_2^{h} \frac{dx}{x(\ln x)^2}$.

don't forget the limit definition. - evaluating an integrated expression at $(\dots)\Big|_2^{\infty}$ doesn't really mean anything

Set $u = \ln x \Rightarrow du = \frac{dx}{x}$. Therefore

evaluate integral separately

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\ln x}.$$

Then

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{h\to\infty} -\frac{1}{\ln x}\Big]_2^{h}$$

$$= \lim_{h\to\infty} \left(-\frac{1}{\ln h} + \frac{1}{\ln 2}\right)$$

now apply bounds on integration and take limit

$$= \frac{1}{\ln 2} < \infty.$$

now conclusion follows

By the integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges

10 3. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n}{3^n(n+1)}(x-2)^n$$

Since the general term is a collection of factors, the ratio test is the best to use here.

good to say, but not actually worth much.

By the ratio test we need

$$\lim_{n\to\infty} \left| \frac{\dfrac{(n+1)(x-2)^{n+1}}{3^{n+1}(n+2)}}{\dfrac{n(x-2)^n}{3^n(n+1)}} \right| < 1$$

clear statement of test. Mess up here and the rest is hell!

$$\Leftrightarrow \lim_{n\to\infty} |x-2| \frac{3^n(n+1)^2}{3^{n+1}\,n\,(n+2)} < 1$$

Some properties of limits

$$\Leftrightarrow \frac{|x-2|}{3} \lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)} < 1$$

$$\Leftrightarrow \frac{|x-2|}{3} \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{1+\frac{2}{n}} < 1$$

specific justific^n of limit found

$$\Leftrightarrow \frac{|x-2|}{3} < 1$$

This is equivalent to

$$-3 < x-2 < 3$$

or

$$-1 < x < 5$$

do this only if asked!

For $x>5$ or $x<-1$ we have divergence.

10 4. Use comparison tests to evaluate the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{2n-1}{\sqrt{n^6+1}}$$

Very important for you, but not worth much by itself!

For comparison tests we need to make an initial guess. Since $\frac{2n-1}{\sqrt{n^6+1}} \approx \frac{2n}{\sqrt{n^6}} = \frac{2n}{n^3} = \frac{2}{n^2}$ for n large, the series should be compared to $\frac{2}{n^2}$, which is a convergent p-series.

this is enough to justify the inequalities

Method 1

larger numerator

$$\frac{2n-1}{\sqrt{n^6+1}} \le \frac{2n}{\sqrt{n^6+1}} \le \frac{2n}{\sqrt{n^6}} = \frac{2}{n^2}.$$

smaller denominator

Since $\sum \frac{2}{n^2} = 2 \underbrace{\sum \frac{1}{n^2} \text{ converges}}$, the original series converges

should have said "a p-series with $p = 2 > 1$"

Method 2 With $a_n = \frac{2n-1}{\sqrt{n^6+1}}$ and $b_n = \frac{1}{n^2}$, make it clear what two sequences are used.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n-1}{\sqrt{n^6+1}}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{(2n-1)n^2}{\sqrt{n^6+1}}$$ simplification

$$= \lim_{n\to\infty} \frac{\cancel{n^3}(2-\frac{1}{n})}{\cancel{n^3}\sqrt{1+\frac{1}{n^6}}}$$ justification of limit found

$$= \frac{2-0}{\sqrt{1+0}} = 2.$$

By limit comparison test, since $\sum \frac{1}{n^2}$ converges, so does the original series.

State test used to draw your conclusion

5. Does the following series converge absolutely or only conditionally?

$$\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2-1}$$

Tells me you know where you are going!

There are potentially two steps here. First we test for absolute convergence. Because there are a lot of factors, use the ratio test:

Sometimes a test doesn't tell you much!

By the ratio test,

$$\rho = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(n+1)}{(n-1)(n+2)}}{\dfrac{(-1)^n \; n}{(n-1)(n+1)}} \right| = \lim_{n\to\infty} \frac{(n+1)(n+1)(n-1)}{n\,n\,(n+2)}$$

$$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{1}{n})(1-\frac{1}{n})}{(1+\frac{2}{n})} = 1.$$

Thus ratio test is inconclusive.

Instead lets use comparison tests with $\frac{1}{n}$:

Comes from rough reasoning that $\frac{n}{n^2-1} \approx \frac{n}{n^2}$ ~~$\frac{1}{n}$~~ $\longrightarrow$ $$\frac{n}{n^2-1} \geq \frac{n}{n^2} = \frac{1}{n}$$

denominator larger — justify ineq

~~Since~~ $\Sigma \frac{1}{n}$ diverges (harmonic), we know $\sum \frac{n}{n^2-1}$ diverges.

Having found that absolute convergence fails, test with the alternating series test:

With $a_n = \frac{n}{n^2-1}$ we see that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} \frac{1}{1-\frac{1}{n^2}} = 0.$

This is correct, but a bit intricate. I would settle for less, but it must be made clear that "decreasing" is needed

Also,

$$a_n - a_{n+1} = \frac{n}{n^2-1} - \frac{n+1}{(n+1)^2-1} = \frac{n}{(n-1)(n+1)} - \frac{n+1}{n^2+2n}$$

$$= \frac{n(n^2+2n) - (n+1)(n^2-1)}{n(n-1)(n+1)(n+2)}$$

$$= \frac{\cancel{n^3} + 2n^2 - \cancel{n^3} + n - n^2 + 1}{n(n-1)(n+1)(n+2)}$$

$$= \frac{n^2+n+1}{(n-1)n(n+1)(n+2)} > 0.$$

Thus the sequence a_n is decreasing. Hence the series converges conditionally.

10 6. Write out the finite Taylor expansion for $f(x)=\sqrt{1+x}$ with $a=0$ and $n=3$. Use this expansion to approximate $\sqrt{1.2}$, and estimate the size of the error.

The first part of the problem is very specific, so we can proceed with it directly:

Since $f(x) = (1+x)^{1/2}$ we have $\longrightarrow f(0) = 1$

$$f'(x) = \tfrac{1}{2}(1+x)^{-1/2} \longrightarrow f'(0) = \tfrac{1}{2}$$
$$f''(x) = -\tfrac{1}{4}(1+x)^{-3/2} \longrightarrow f''(0) = -\tfrac{1}{4}$$
$$f'''(x) = \tfrac{3}{8}(1+x)^{-5/2} \longrightarrow f'''(0) = \tfrac{3}{8}$$
$$f^{iv}(x) = -\tfrac{15}{16}(1+x)^{-7/2} \longrightarrow f^{iv}(z) = -\frac{15}{16(1+z)^{7/2}}.$$

Organize your calculations so I don't have to search for them.

beautifully clear statement of the result

Thus
$$\sqrt{1+x} = \underbrace{1 + \frac{1}{2}x - \frac{1}{4\cdot 2!}x^2 + \frac{3}{8\cdot 3!}x^3}_{P_3(x)} + \underbrace{\left(-\frac{15}{16\cdot 4!}\right)\frac{1}{(1+z)^{7/2}}x^4}_{R_3(x)} \quad \text{where } 0<z<x.$$

ABSOLUTELY MUST BE THERE!

By direct calculation,
$$P_3(.2) = 1 + \frac{.2}{2} - \frac{(.2)^2}{4\cdot 2!} + \frac{3(.2)^3}{8\cdot 3!} = 1.0955$$

To estimate the error we need to find an upper bound on $R_3(.2)$. Use some experimentation, especially with the endpoints of $0<z<.2$.

Clearly $\dfrac{1}{(1+z)^{7/2}} \le \dfrac{1}{(1+0)^{7/2}}$ for $0 \le z \le .2$, so

$$|R_3(x)| \le \frac{15}{16\cdot 4!}(.2)^4 = .00006$$

should have said "smaller denom." to justify