Math 132 — **Test No. 3. - Solutions** — R. Muncaster

1\. a) The squeeze rule states that if $\{a_n\}, \{b_n\}, \{c_n\}$ are three sequences such that $a_n \leq b_n \leq c_n$ for all n and

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$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$.

b)(i) Note that

$$-\frac{n^{2/3}}{n+1} \leq \frac{n^{2/3}\sin(n!)}{n+1} \leq \frac{n^{2/3}}{n+1}.$$

(6) But $\displaystyle\lim_{n\to\infty} \frac{n^{2/3}}{n+1} = \lim_{n\to\infty} \frac{1}{n^{1/3}+n^{-2/3}} = 0$

By the squeeze rule, $\displaystyle\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$

(ii) $\displaystyle\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = \lim_{n\to\infty} \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$

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2\. We consider

$$\int_2^\infty \frac{dx}{x(\ln x)^s} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^s}.$$

Let $\ln x = y$. Then

$$\int \frac{dx}{x(\ln x)^s} = \int \frac{dy}{y^s} = \begin{cases} \ln y & \text{if } s=1 \\ \dfrac{y^{-s+1}}{-s+1} & \text{if } s\neq 1. \end{cases}$$

Hence, if $s=1$,

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$$\int_2^\infty \frac{dx}{x(\ln x)^s} = \lim_{b\to\infty}\left[\ln(\ln b) - \ln(\ln 2)\right] = \infty.$$

If $s<1$, then

$$\int_2^\infty \frac{dx}{x(\ln x)^s} = \lim_{b\to\infty}\left[\frac{(\ln b)^{1-s}}{1-s} - \frac{(\ln 2)^{1-s}}{1-s}\right] = \infty.$$

If $s>1$, then

$$\int_2^\infty \frac{dx}{x(\ln x)^s} = \lim_{b\to\infty}\left[\frac{1}{(1-s)(\ln b)^{s-1}} - \frac{1}{(1-s)(\ln 2)^{s-1}}\right] = -\frac{1}{(1-s)(\ln 2)^{s-1}} \neq \infty.$$

Thus, $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^s}$ converges if $s>1$ and diverges if $0<s\leq 1$.

3. a) By the root test,

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$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$ Thus $\sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$ converge because $\rho < 1$.

b) By the ratio test

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n\, n!}$$

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$$= \lim_{n\to\infty} \frac{2\cancel{(n+1)}\, n^n}{(n+1)^n\, \cancel{(n+1)}}$$

$$= 2 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \frac{2}{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}$$

$$= \frac{2}{e}$$

But $e > 2$, so $\rho < 1$. Thus $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ converges.

4. a) Set $a_n = \frac{\sqrt{n}}{n+100}$. Then set $f(x) = \frac{\sqrt{x}}{x+100}$. We see that

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}}{x+100} - \frac{\sqrt{x}}{(x+100)^2} = \frac{x+100 - 2x}{2\sqrt{x}\,(x+100)^2} = \frac{100-x}{2\sqrt{x}\,(x+100)^2} < 0 \text{ for } x > 100.$$

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This shows that

$a_n \geqq a_{n+1}$ for all $n > 100$. Also, $\lim_{n\to\infty} a_n = 0$.

By the theorem on alternating series, we conclude the series converges.

b) $$\frac{1+\sqrt{n}}{(n+1)^3-1} = \frac{1+\sqrt{n}}{n^3+3n^2+3n} \leqq \frac{2\sqrt{n}}{n^3} = \frac{2}{n^{5/2}}.$$ Since $\sum \frac{2}{n^{5/2}}$ converges, we

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conclude that $\sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{(n+1)^3-1}$ converges.

c) $$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{e^{n^2}}{e^{(n+1)^2}} = \lim_{n\to\infty} \frac{e^{n^2}}{e^{n^2+2n+1}} = \lim_{n\to\infty} \frac{1}{e^{2n+1}} = 0$$

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Thus, since $\rho < 1$, we conclude that $\sum_{n=1}^{\infty} e^{-n^2}$ converges.

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