

## 6.4

## Areas of Surfaces of Revolution

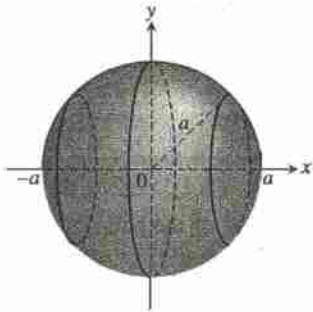


FIGURE 6.27 Rotating the semicircle  $y = \sqrt{a^2 - x^2}$  of radius  $a$  with center at the origin generates a spherical surface with area  $4\pi a^2$ .

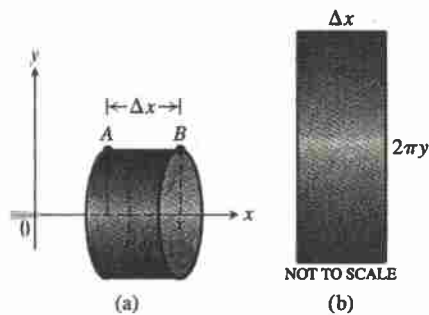


FIGURE 6.28 (a) A cylindrical surface generated by rotating the horizontal line segment  $AB$  of length  $\Delta x$  about the  $x$ -axis has area  $2\pi y \Delta x$ . (b) The cut and rolled out cylindrical surface as a rectangle.

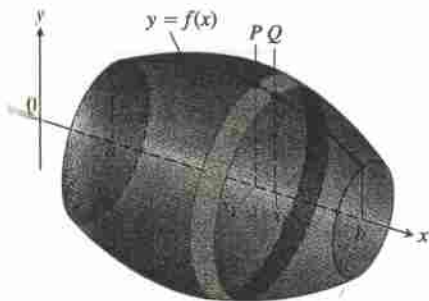


FIGURE 6.30 The surface generated by revolving the graph of a nonnegative function  $y = f(x)$ ,  $a \leq x \leq b$ , about the  $x$ -axis. The surface is a union of bands like the one swept out by the arc  $PQ$ .

When you jump rope, the rope sweeps out a surface in the space around you similar to what is called a *surface of revolution*. The “area” of this surface depends on the length of the rope and the distance of each of its segments from the axis of revolution. In this section we define areas of surfaces of revolution. More complicated surfaces are treated in Chapter 14.

## Defining Surface Area

We want our definition of the area of a surface of revolution to be consistent with known results from classical geometry for the surface areas of spheres, circular cylinders, and cones. So if the jump rope discussed in the introduction takes the shape of a semicircle with radius  $a$  rotated about the  $x$ -axis (Figure 6.27), it generates a sphere with surface area  $4\pi a^2$ .

Before considering general curves, we begin by rotating horizontal and slanted line segments about the  $x$ -axis. If we rotate the horizontal line segment  $AB$  having length  $\Delta x$  about the  $x$ -axis (Figure 6.28a), we generate a cylinder with surface area  $2\pi y \Delta x$ . This area is the same as that of a rectangle with side lengths  $\Delta x$  and  $2\pi y$  (Figure 6.28b). The length  $2\pi y$  is the circumference of the circle of radius  $y$  generated by rotating the point  $(x, y)$  on the line  $AB$  about the  $x$ -axis.

Suppose the line segment  $AB$  has length  $\Delta s$  and is slanted rather than horizontal. Now when  $AB$  is rotated about the  $x$ -axis, it generates a frustum of a cone (Figure 6.29a). From classical geometry, the surface area of this frustum is  $2\pi y^* \Delta s$ , where  $y^* = (y_1 + y_2)/2$  is the average height of the slanted segment  $AB$  above the  $x$ -axis. This surface area is the same as that of a rectangle with side lengths  $\Delta s$  and  $2\pi y^*$  (Figure 6.29b).

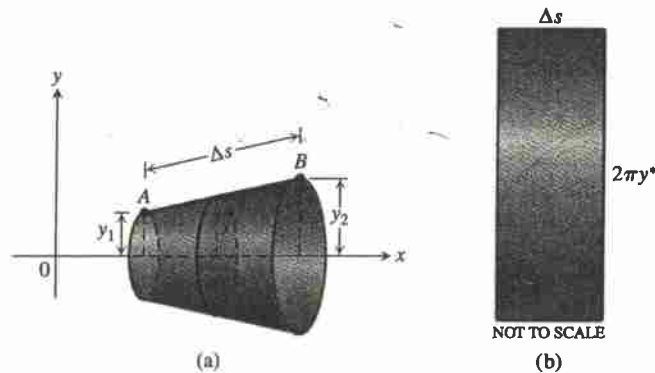


FIGURE 6.29 (a) The frustum of a cone generated by rotating the slanted line segment  $AB$  of length  $\Delta s$  about the  $x$ -axis has area  $2\pi y^* \Delta s$ . (b) The area of the rectangle for  $y^* = \frac{y_1 + y_2}{2}$ , the average height of  $AB$  above the  $x$ -axis.

Let's build on these geometric principles to define the area of a surface swept out by revolving more general curves about the  $x$ -axis. Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative continuous function  $y = f(x)$ ,  $a \leq x \leq b$ , about the  $x$ -axis. We partition the closed interval  $[a, b]$  in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.30 shows a typical arc  $PQ$  and the band it sweeps out as part of the graph of  $f$ .

As the arc  $PQ$  revolves about the  $x$ -axis, the line segment joining  $P$  and  $Q$  sweeps out a frustum of a cone whose axis lies along the  $x$ -axis (Figure 6.31). The surface area

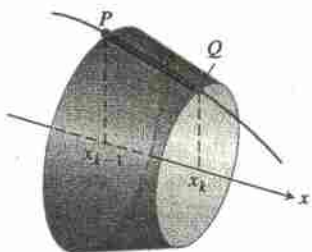


FIGURE 6.31 The line segment joining  $P$  and  $Q$  sweeps out a frustum of a cone.

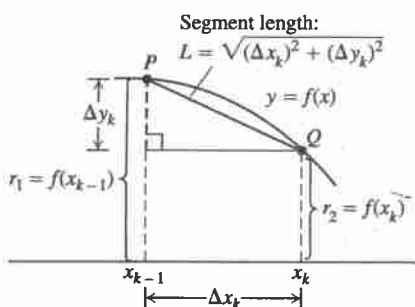


FIGURE 6.32 Dimensions associated with the arc and line segment  $PQ$ .

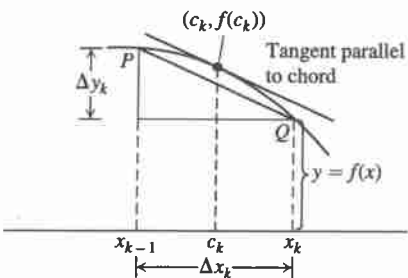


FIGURE 6.33 If  $f$  is smooth, the Mean Value Theorem guarantees the existence of a point  $c_k$  where the tangent is parallel to segment  $PQ$ .

of this frustum approximates the surface area of the band swept out by the arc  $PQ$ . The surface area of the frustum of the cone shown in Figure 6.31 is  $2\pi y^*L$ , where  $y^*$  is the average height of the line segment joining  $P$  and  $Q$ , and  $L$  is its length (just as before). Since  $f \geq 0$ , from Figure 6.32 we see that the average height of the line segment is  $y^* = (f(x_{k-1}) + f(x_k))/2$ , and the slant length is  $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$ . Therefore,

$$\begin{aligned} \text{Frustum surface area} &= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \end{aligned}$$

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc  $PQ$ , is approximated by the frustum area sum

$$\sum_{k=1}^n \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \quad (1)$$

We expect the approximation to improve as the partition of  $[a, b]$  becomes finer. Moreover, if the function  $f$  is differentiable, then by the Mean Value Theorem, there is a point  $(c_k, f(c_k))$  on the curve between  $P$  and  $Q$  where the tangent is parallel to the segment  $PQ$  (Figure 6.33). At this point,

$$\begin{aligned} f'(c_k) &= \frac{\Delta y_k}{\Delta x_k}, \\ \Delta y_k &= f'(c_k) \Delta x_k. \end{aligned}$$

With this substitution for  $\Delta y_k$ , the sums in Equation (1) take the form

$$\begin{aligned} &\sum_{k=1}^n \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (f'(c_k) \Delta x_k)^2} \\ &= \sum_{k=1}^n \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2} \Delta x_k. \end{aligned} \quad (2)$$

These sums are not the Riemann sums of any function because the points  $x_{k-1}$ ,  $x_k$ , and  $c_k$  are not the same. However, it can be proved that as the norm of the partition of  $[a, b]$  goes to zero, the sums in Equation (2) converge to the integral

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} dx.$$

We therefore define this integral to be the area of the surface swept out by the graph of  $f$  from  $a$  to  $b$ .

**DEFINITION** If the function  $f(x) \geq 0$  is continuously differentiable on  $[a, b]$ , the area of the surface generated by revolving the curve  $y = f(x)$  about the  $x$ -axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} dx. \quad (3)$$

The square root in Equation (3) is the same one that appears in the formula for the length of the generating curve in Equation (2) of Section 6.3.

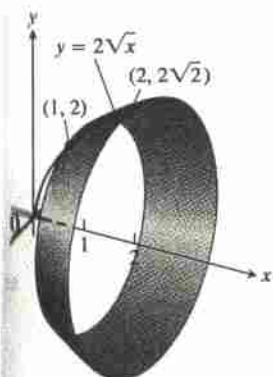


FIGURE 6.34 In Example 1 we calculate the area of this surface.

**EXAMPLE 1** Find the area of the surface generated by revolving the curve  $y = 2\sqrt{x}$ ,  $1 \leq x \leq 2$ , about the  $x$ -axis (Figure 6.34).

**Solution** We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{Eq. (3)}$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\begin{aligned} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} \\ &= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}. \end{aligned}$$

With these substitutions,

$$\begin{aligned} S &= \int_1^2 2\pi \cdot 2\sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}} dx = 4\pi \int_1^2 \sqrt{x+1} dx \\ &= 4\pi \cdot \frac{2}{3} (x+1)^{3/2} \Big|_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}). \end{aligned}$$

### Revolution About the $y$ -Axis

For revolution about the  $y$ -axis, we interchange  $x$  and  $y$  in Equation (3).

#### Surface Area for Revolution About the $y$ -Axis

If  $x = g(y) \geq 0$  is continuously differentiable on  $[c, d]$ , the area of the surface generated by revolving the curve  $x = g(y)$  about the  $y$ -axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy. \quad (4)$$

**EXAMPLE 2** The line segment  $x = 1 - y$ ,  $0 \leq y \leq 1$ , is revolved about the  $y$ -axis to generate the cone in Figure 6.35. Find its lateral surface area (which excludes the base area).

**Solution** Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

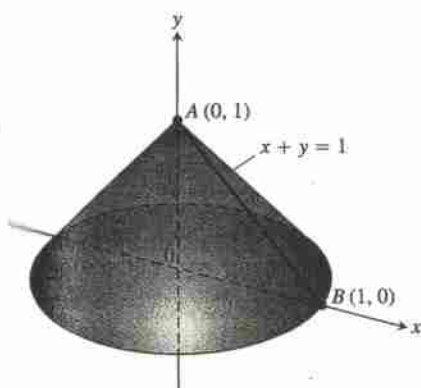


FIGURE 6.35 Revolving line segment  $AB$  about the  $y$ -axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

and calculate

$$\begin{aligned} S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 2\pi(1-y)\sqrt{2} dy \\ &= 2\pi\sqrt{2} \left[ y - \frac{y^2}{2} \right]_0^1 = 2\pi\sqrt{2} \left( 1 - \frac{1}{2} \right) \\ &= \pi\sqrt{2}. \end{aligned}$$

The results agree, as they should. ■

### Parametrized Curves

Regardless of the coordinate axis of revolution, the square roots appearing in Equations (3) and (4) are the same ones that appear in the formulas for arc length in Section 6.3. If the curve is parametrized by the equations  $x = f(t)$  and  $y = g(t)$ ,  $a \leq t \leq b$ , where  $f$  and  $g$  are continuously differentiable and  $(f')^2 + (g')^2 > 0$  on  $[a, b]$ , then the corresponding square root appearing in the arc length formula is

$$\sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

This observation leads to the following formulas for area of surfaces of revolution for smooth parametrized curves.

#### Surface Area of Revolution for Parametrized Curves

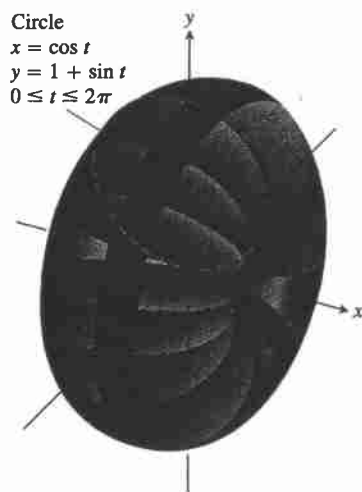
If a smooth curve  $x = f(t)$ ,  $y = g(t)$ ,  $a \leq t \leq b$ , is traversed exactly once as  $t$  increases from  $a$  to  $b$ , then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the  $x$ -axis ( $y \geq 0$ ):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (5)$$

2. Revolution about the  $y$ -axis ( $x \geq 0$ ):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (6)$$



**FIGURE 6.36** In Example 3 we calculate the area of the surface of revolution swept out by this parametrized curve.

As with length, we can calculate surface area from any convenient parametrization that meets the stated criteria.

**EXAMPLE 3** The standard parametrization of the circle of radius 1 centered at the point  $(0, 1)$  in the  $xy$ -plane is

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \leq t \leq 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the  $x$ -axis (Figure 6.36).

**Solution** We evaluate the formula

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{2\pi} 2\pi(1 + \sin t) \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_1} dt \\ &= 2\pi \int_0^{2\pi} (1 + \sin t) dt \\ &= 2\pi [t - \cos t]_0^{2\pi} = 4\pi^2. \end{aligned}$$

Eq. (5) for revolution about the  $x$ -axis;  
 $y = 1 + \sin t > 0$

## EXERCISES 6.4

In Exercises 1–8:

- Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.
- T** Graph the curve to see what it looks like. If you can, graph the surface, too.
- T** Use your grapher's or computer's integral evaluator to find the surface's area numerically.

- $y = \tan x$ ,  $0 \leq x \leq \pi/4$ ;  $x$ -axis
- $y = x^2$ ,  $0 \leq x \leq 2$ ;  $x$ -axis
- $xy = 1$ ,  $1 \leq y \leq 2$ ;  $y$ -axis
- $x = \sin y$ ,  $0 \leq y \leq \pi$ ;  $y$ -axis
- $x^{1/2} + y^{1/2} = 3$  from  $(4, 1)$  to  $(1, 4)$ ;  $x$ -axis
- $y + 2\sqrt{y} = x$ ,  $1 \leq y \leq 2$ ;  $y$ -axis
- $x = \int_0^y \tan t dt$ ,  $0 \leq y \leq \pi/3$ ;  $y$ -axis
- $y = \int_1^x \sqrt{t^2 - 1} dt$ ,  $1 \leq x \leq \sqrt{5}$ ;  $x$ -axis

- Find the lateral (side) surface area of the cone generated by revolving the line segment  $y = x/2$ ,  $0 \leq x \leq 4$ , about the  $x$ -axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

- Find the lateral surface area of the cone generated by revolving the line segment  $y = x/2$ ,  $0 \leq x \leq 4$  about the  $y$ -axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

- Find the surface area of the cone frustum generated by revolving the line segment  $y = (x/2) + (1/2)$ ,  $1 \leq x \leq 3$ , about the  $x$ -axis. Check your result with the geometry formula

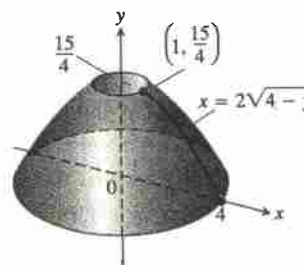
$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

- Find the surface area of the cone frustum generated by revolving the line segment  $y = (x/2) + (1/2)$ ,  $1 \leq x \leq 3$ , about the  $y$ -axis. Check your result with the geometry formula

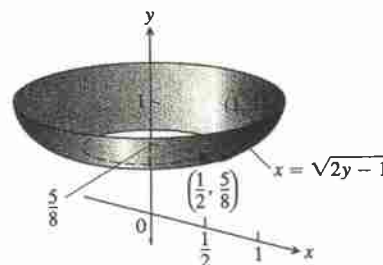
$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

Find the areas of the surfaces generated by revolving the curves in Exercises 13–23 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

- $y = x^3/9$ ,  $0 \leq x \leq 2$ ;  $x$ -axis
- $y = \sqrt{x}$ ,  $3/4 \leq x \leq 15/4$ ;  $x$ -axis
- $y = \sqrt{2x - x^2}$ ,  $0.5 \leq x \leq 1.5$ ;  $x$ -axis
- $y = \sqrt{x + 1}$ ,  $1 \leq x \leq 5$ ;  $x$ -axis
- $x = y^3/3$ ,  $0 \leq y \leq 1$ ;  $y$ -axis
- $x = (1/3)y^{3/2} - y^{1/2}$ ,  $1 \leq y \leq 3$ ;  $y$ -axis
- $x = 2\sqrt{4 - y}$ ,  $0 \leq y \leq 15/4$ ;  $y$ -axis



- $x = \sqrt{2y - 1}$ ,  $5/8 \leq y \leq 1$ ;  $y$ -axis



## Selected Problems Answers

### Section 6.4, pp. 419–421

1. (a)  $2\pi \int_0^{\pi/4} \tan x \sqrt{1 + \sec^4 x} dx$  (c)  $\approx 3.84$

3. (a)  $2\pi \int_1^2 \frac{1}{y} \sqrt{1 + y^{-4}} dy$  (c)  $\approx 5.02$

5. (a)  $2\pi \int_1^4 (3 - \sqrt{x})^2 \sqrt{1 + (1 - 3x^{-1/2})^2} dx$  (c)  $\approx 63.37$

7. (a)  $2\pi \int_0^{\pi/3} \left( \int_0^y \tan t dt \right) \sec y dy$  (c)  $\approx 2.08$

9.  $4\pi\sqrt{5}$  11.  $3\pi\sqrt{5}$  13.  $98\pi/81$  15.  $2\pi$

17.  $\pi(\sqrt{8} - 1)/9$  19.  $35\pi\sqrt{5}/3$  21.  $\pi\left(\frac{15}{16} + \ln 2\right)$

23.  $253\pi/20$  27. Order 226.2 liters of each color.

33.  $8\pi^2$  35.  $52\pi/3$

37.  $\frac{16}{3}\pi(e^{3/2} + 3e^{1/2} - 4)$  39.  $3\pi\sqrt{5}$