

6.4

Areas of Surfaces of Revolution

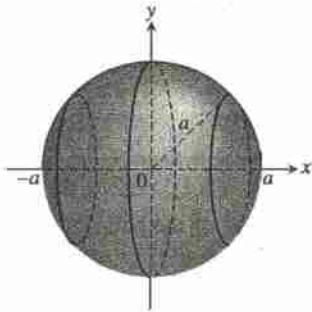


FIGURE 6.27 Rotating the semicircle $y = \sqrt{a^2 - x^2}$ of radius a with center at the origin generates a spherical surface with area $4\pi a^2$.

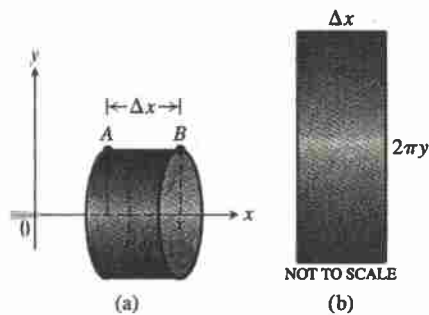


FIGURE 6.28 (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x -axis has area $2\pi y \Delta x$. (b) The cut and rolled out cylindrical surface as a rectangle.

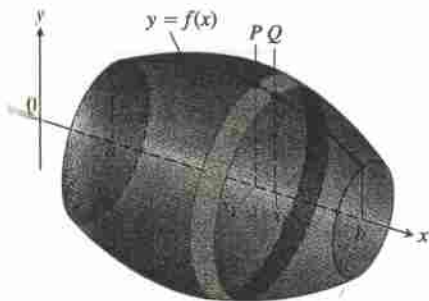


FIGURE 6.30 The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \leq x \leq b$, about the x -axis. The surface is a union of bands like the one swept out by the arc PQ .

When you jump rope, the rope sweeps out a surface in the space around you similar to what is called a *surface of revolution*. The “area” of this surface depends on the length of the rope and the distance of each of its segments from the axis of revolution. In this section we define areas of surfaces of revolution. More complicated surfaces are treated in Chapter 14.

Defining Surface Area

We want our definition of the area of a surface of revolution to be consistent with known results from classical geometry for the surface areas of spheres, circular cylinders, and cones. So if the jump rope discussed in the introduction takes the shape of a semicircle with radius a rotated about the x -axis (Figure 6.27), it generates a sphere with surface area $4\pi a^2$.

Before considering general curves, we begin by rotating horizontal and slanted line segments about the x -axis. If we rotate the horizontal line segment AB having length Δx about the x -axis (Figure 6.28a), we generate a cylinder with surface area $2\pi y \Delta x$. This area is the same as that of a rectangle with side lengths Δx and $2\pi y$ (Figure 6.28b). The length $2\pi y$ is the circumference of the circle of radius y generated by rotating the point (x, y) on the line AB about the x -axis.

Suppose the line segment AB has length Δs and is slanted rather than horizontal. Now when AB is rotated about the x -axis, it generates a frustum of a cone (Figure 6.29a). From classical geometry, the surface area of this frustum is $2\pi y^* \Delta s$, where $y^* = (y_1 + y_2)/2$ is the average height of the slanted segment AB above the x -axis. This surface area is the same as that of a rectangle with side lengths Δs and $2\pi y^*$ (Figure 6.29b).

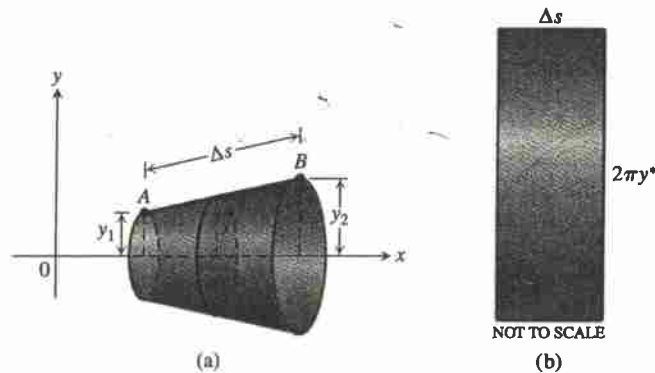


FIGURE 6.29 (a) The frustum of a cone generated by rotating the slanted line segment AB of length Δs about the x -axis has area $2\pi y^* \Delta s$. (b) The area of the rectangle for $y^* = \frac{y_1 + y_2}{2}$, the average height of AB above the x -axis.

Let’s build on these geometric principles to define the area of a surface swept out by revolving more general curves about the x -axis. Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative continuous function $y = f(x)$, $a \leq x \leq b$, about the x -axis. We partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.30 shows a typical arc PQ and the band it sweeps out as part of the graph of f .

As the arc PQ revolves about the x -axis, the line segment joining P and Q sweeps out a frustum of a cone whose axis lies along the x -axis (Figure 6.31). The surface area