Math 285 - Differential Equations
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Instructions: *You have 3 hours to complete your work. Notes, texts, and other aids are not permitted in the examination. Calculators may be used, but are not necessary. Complete answers are required for full credit.*

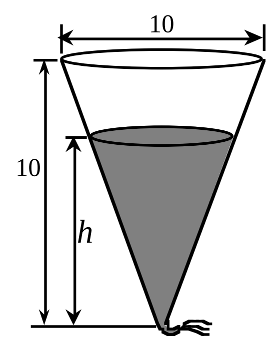
1. (10 points) Find the unique solution of the initial value problem

$$(3x^2 + 2xe^{4y}) + 4x^2e^{4y}y' = 0, \quad y(2) = 0,$$

and explicitly find the value $y(1)$.

2. (10 points) The conical shaped water tank at the right is slowly losing water at its lower end. As a result, the height of the water in the tank, given by $h(t)$, is decreasing according to the rule

$$\frac{dh}{dt} = -\frac{4\sqrt{2g}}{\pi h^{3/2}}$$



a) Solve this differential equation for $h(t)$ for the general initial depth

$$h(0) = h_0$$

b) If the tank is initially full, how long does it take for the tank to empty?

3. (10 points) Solve the initial value problem

$$y''' - y'' + 9y' - 9y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 30.$$

4. (10 points) Find the general solution of the non-homogeneous equation

$$y'' - 5y' + 6y = (x+1)e^{2x}.$$

5. (10 points) Find a particular solution of the differential equation

$$(x-1)y'' - xy' + y = -(x-1)^2,$$

given that x and e^x are solutions of the associated homogeneous problem.

6. (10 points) Find two linearly independent power series solutions of the equation

$$(1 - x^2)y'' + 3xy' + 5y = 0.$$

You need compute only the first three non-zero terms in each series.

7. (10 points) For the function $f(x) = x^2, \quad 0 \le x \le \pi$,

a) find the Fourier cosine series of f, and

b) what is the value of this cosine series at $x = \pi$?

8. (10 points) Find all separated solutions $u(x,y) = P(x)Q(y)$ of the following problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \begin{cases} 0 < x < 2 \\ 0 < y < 2 \end{cases}$$
$$u(x,0) = 0, \quad u(x,2) = 0,$$
$$u(0,y) = 0.$$

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Useful Facts

Boundary Value Problem 1-

For the two point boundary value problem

$$y'' + \sigma y = 0, \quad y(0) = 0, \quad y(\ell) = 0,$$

the only non-zero solutions y, together with the values σ that produce them, are

$$\sigma = \frac{n^2\pi^2}{\ell^2}, \quad y(x) = \sin\frac{n\pi x}{l}, \quad \text{for } n = 1,2,3,\ldots$$

Boundary Value Problem 2-

For the two point boundary value problem

$$y'' + \sigma y = 0, \quad y'(0) = 0, \quad y'(\ell) = 0,$$

the only non-zero solutions y, together with the values σ that produce them, are

$\sigma = 0, \quad y(x) = 1,$ and

$$\sigma = \frac{n^2\pi^2}{\ell^2}, \quad y(x) = \cos\frac{n\pi x}{l}, \quad \text{for } n = 1,2,3,\ldots$$