

## Math 415 - Assignment 1 Solutions

Problems: 1.2.8, 1.2.11 (with explanations), 1.2.14, 1.2.32, 1.3.3 (e), 1.3.3 (g), 1.3.8, 1.3.10, 1.3.13, 1.3.14, 1.3.22 (g), 1.4.3 (c), 1.4.9, 1.4.15, 1.4.20 (c), 1.5.4, 1.5.5, 1.5.16, 1.5.32 (d)

Problem 1.2.8

(a) No, the matrices do not commute:

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 3 \\ 1 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ 5 & 10 \end{pmatrix}$$

(b) No. One product is a 2x2 matrix and the other product is a 3x3 matrix.

(c) Yes, these matrices commute!

$$\begin{pmatrix} 3 & 0 & -1 \\ -2 & -1 & 2 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -2 \\ -1 & -1 & 1 \\ 4 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ -2 & -1 & 2 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -2 \\ -1 & -1 & 1 \\ 4 & 0 & -2 \end{pmatrix}$$

Problem 1.2.11

(a) True. Illustrate in a 3x3 case:

$$\begin{pmatrix} a & 0 & -0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} A & 0 & -0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} = \begin{pmatrix} aA & 0 & 0 \\ 0 & bB & 0 \\ 0 & 0 & cC \end{pmatrix}$$

here the diagonal entries are just the products of the diagonal entries of the matrices being multiplied.

(b) True. Each off diagonal term in the product comes from a row and a column in which the single non-zero entry are in different positions.

Problem 1.2.14

Let  $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$  commute with  $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ . Then  $AX = XA$ . That is

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 2x+y \\ z & 2z+w \end{pmatrix} = \begin{pmatrix} x+2z & y+2w \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

So we need to solve the four equations  $x = x + 2z$ ,  $2x + y = y + 2w$ ,  $z = z$ ,  $2z + w = w$ . These show that  $z = 0$  and  $x = w$ . So all matrices that commute with  $A$  have the form

$$X = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$$

for arbitrary values of  $x$  and  $y$ .

Problem 1.2.32

$$(a) \operatorname{tr} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = 1 + 3 = 4, \operatorname{tr} \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \\ -4 & 3 & -1 \end{pmatrix} = 1 + 0 + (-1) = 0$$

(b) Illustrate for 3x3

$$\operatorname{tr} \left( \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} + \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \right) = \operatorname{tr} \begin{pmatrix} a+A & b+B & c+C \\ d+D & e+E & f+F \\ g+G & h+H & i+I \end{pmatrix}$$

$$= a + A + b + B + c + c = (a + b + c) + (A + B + C) = \operatorname{tr} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} + \operatorname{tr} \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

$$(c) \operatorname{tr} \left( \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \right) = \operatorname{tr} \begin{pmatrix} aA + bD + cG & aB + bE + cH & aC + bF + cI \\ dA + eD + fG & dB + eE + fH & dC + eF + fI \\ gA + hD + iG & gB + hE + iH & gC + hF + iI \end{pmatrix}$$

$$= (aA + bD + cG) + (dB + eE + fH) + (gC + hF + iI) \text{ and}$$

$$\begin{aligned} \text{tr} \left( \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right) &= \text{tr} \begin{pmatrix} aA + dB + gC & Ab + Be + Ch & Ac + Bf + iC \\ Da + Ed + Fg & bD + eE + hF & Dc + Ef + iF \\ Ga + Hd + Ig & Gb + He + Ih & cG + fH + iI \end{pmatrix} \\ &= (aA + dB + gC) + (bD + eE + hF) + (cG + fH + iI) \\ &= (aA + bD + cG) + (dB + eE + fH) + (gC + hF + iI). \text{ Done!} \end{aligned}$$

Another way to reason is, if  $A = (a_{ij})$  and  $B = (b_{ij})$ , then  $AB = (\sum_{k=1}^n a_{ik}b_{kj})$  and  $BA = (\sum_{k=1}^n b_{ik}a_{kj})$

Therefore  $\text{tr}AB = \sum_{i=1}^n \sum_{k=1}^n a_{ik}b_{ki}$  and  $\text{tr}BA = \sum_{i=1}^n \sum_{k=1}^n b_{ik}a_{ki}$ . Since  $i$  and  $k$  in these two sums are dummy variables, we can interchange  $i$  and  $k$  in the first sum to obtain the second sum.

(d) by parts (b) and (c),  $\text{tr}(AB - BA) = \text{tr}AB - \text{tr}BA = \text{tr}AB - \text{tr}AB = 0$

(e) No unless  $m = n$ . In general  $AB$  will be  $m \times m$  and  $BA$  will be  $n \times n$ .

Problem 1.3.3 (e)

By Gaussian Elimination: 
$$\left( \begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 3 & 3 \\ -1 & -1 & 3 & 5 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 5 & 3 \\ 0 & 0 & 2 & 5 \end{array} \right)$$

and then back substitution now involves the equations

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 5 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}, \text{ i.e. } 2r = 5, -3q + 5r = 3, p + q - r = 0$$

These lead to the solution  $r = \frac{5}{2}, q = \frac{3}{2}, r = 1$ .

Problem 1.3.3 (g)

By Gaussian Elimination: 
$$\left( \begin{array}{cccc|c} 2 & -3 & 1 & 1 & -1 \\ 1 & -1 & -2 & -1 & 0 \\ 3 & -2 & 1 & 2 & 5 \\ 1 & 3 & 2 & 1 & 3 \end{array} \right) \rightarrow \left( \begin{array}{cccc|c} 2 & -3 & 1 & 1 & -1 \\ 0 & \frac{1}{2} & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{13}{2} \\ 0 & \frac{9}{2} & \frac{3}{2} & \frac{1}{2} & \frac{7}{2} \end{array} \right) \rightarrow$$

$$\left( \begin{array}{cccc|c} 2 & -3 & 1 & 1 & -1 \\ 0 & \frac{1}{2} & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 12 & 8 & 4 \\ 0 & 0 & 24 & 14 & -1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|c} 2 & -3 & 1 & 1 & -1 \\ 0 & \frac{1}{2} & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 12 & 8 & 4 \\ 0 & 0 & 0 & -2 & -9 \end{array} \right)$$

and then by back substitution we have the equation  $-2w = -9$ , i.e.  $w = \frac{9}{2}$ . Then  $12z + 8w = 4$ , so  $z = \frac{1}{12}(4 - \frac{72}{2}) = -\frac{8}{3}$ . Then  $\frac{1}{2}y - \frac{5}{2}z - \frac{3}{2}w = \frac{1}{2}$ , so  $y = 2(\frac{1}{2} + \frac{5}{2}(-\frac{8}{3}) + \frac{3}{2}\frac{9}{2}) = \frac{7}{6}$ . Finally,  $2x - 3y + z + w = -1$ , so  $x = \frac{1}{2}(-1 + 3(\frac{7}{6}) - (-\frac{8}{3}) - \frac{9}{2}) = \frac{1}{3}$

Problem 1.3.8

One solution of  $Ax = 0$  is certainly  $x = 0$ . Then we use the fact that a *regular* square system has just one solution. It must therefore be  $x = 0$ .

Problem 1.3.10  

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} aA & aB + bC \\ 0 & cC \end{pmatrix} = \begin{pmatrix} aA & Ab + Bc \\ 0 & cC \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$
 and so these two commute if and only if  $aB + bC = Ab + Bc$ .

Problem 1.3.13  
 (a) 
$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Illustrate with the 4x4 case. Strictly upper triangular means that the diagonal entries are zero. Thus

$$\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ad & ae + bf \\ 0 & 0 & 0 & df \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ Notice that the diagonal of zeros}$$

has moved up one place to the first *upper* off-diagonal. Let's try it again:

$$\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & adf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ Now the diagonal of}$$

zeros has moved up to the *second* off-diagonal. Clearly, enough multiplications will move the zeros up far enough that the final product is the zero matrix, i.e. the original matrix is *nilpotent*.

$$(c) \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for any } a \text{ and } b.$$

Problem 1.3.14 In each of these cases, ask yourself what elementary row operation you would have to apply to the identity matrix to produce the given matrix.

(a) Add (-2) times row 2 to row 1. Here is an illustration:

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - 2c & b - 2d \\ c & d \end{pmatrix}$$

(b) Add 7 times row 1 to row 2

(c) Add (-5) times row 3 to row 2. Here is an illustration:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d - 5g & e - 5h & f - 5i \\ g & h & i \end{pmatrix}$$

(d) Add 1/2 times row 1 to row 3

(e) Add (-3) times row 4 to row 2

Problem 1.3.22 (g)

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & 3 & 0 & 2 \\ 0 & -1 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 3 & -1 & 2 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & -10 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & \frac{3}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & \frac{3}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 3 & 1 \end{pmatrix} = L$$

Here is a verification:

$$LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & \frac{3}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & -10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & 3 & 0 & 2 \\ 0 & -1 & 2 & 1 \end{pmatrix} = A$$

Problem 1.4.3 (c)

$$\left( \begin{array}{ccc|c} 0 & 1 & -1 & 4 \\ -2 & -5 & 0 & 2 \\ 1 & 0 & 1 & -8 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 1 & -8 \\ -2 & -5 & 0 & 2 \\ 0 & 1 & -1 & 4 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 1 & -8 \\ 0 & -5 & 2 & -14 \\ 0 & 1 & -1 & 4 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 1 & -8 \\ 0 & -5 & 2 & -14 \\ 0 & 0 & -\frac{3}{5} & \frac{6}{5} \end{array} \right)$$

Now back substitution gives us  $x_3 = -2$ , then  $-5x_2 + 2x_3 = -14$ , i.e.  $x_2 = -\frac{1}{5}(-14 + 4) = 2$ , and finally  $x_1 + x_3 = -8$ , i.e.  $x_1 = -8 - (-2) = -6$ .

Problem 1.4.9

$$(a) P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}:$$

$$(b) P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$(c) P_1 P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P_2 P_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

so  $P_1$  and  $P_2$  do not commute.

(d) Ask yourself: What do  $P_1 P_2$  and  $P_2 P_1$  do to the 4x4 identity?  $P_1 P_2$  changes row 4 to row 1, row 1 to row 2, and row 2 to row 4.  $P_2 P_1$  changes row 2 to row 1, row 4 to row 2, and row 1 to row 4.

Problem 1.4.15

$$(a) P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(b) AP = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c & b & a \\ f & e & d \\ i & h & g \end{pmatrix} \text{ which has columns 1 and 3 interchanged.}$$

(c) Yes. Enough to illustrate with the 4x4 case:

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ l & m & n & p \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c & b & a & d \\ g & f & e & h \\ k & j & i & l \\ n & m & l & p \end{pmatrix}$$

Problem 1.4.20 (c)

$$A: \begin{pmatrix} 1 & -1 & 2 & 1 \\ -1 & 1 & -3 & 0 \\ 1 & -1 & 4 & -3 \\ 1 & 2 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -4 \\ 0 & 3 & -3 & 0 \end{pmatrix} \begin{array}{l} \text{add (1) row 1 to row 2} \\ \text{add (-1) row 1 to row 3} \\ \text{add (-1) row 1 to row 4} \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & -1 & 1 \end{pmatrix} \text{ switch rows 2 and 4} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ add } (\frac{1}{2}) \text{ row 3 to row 4}$$

$$= U \begin{pmatrix} 1 & -1 & 1 & -3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix} \text{ add } (-\frac{5}{2}) \text{ row 3 to row 4} = U$$

$$L: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \text{'add (1) row 1 to row 2'} \\ \text{'add (-1) row 1 to row 3'} \\ \text{'add (-1) row 1 to row 4'} \end{array} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \text{ 'switch rows 2 and 4' } \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -\frac{1}{2} & 1 \end{pmatrix} \text{ 'add } (\frac{1}{2}) \text{ row 3 to row 4' } = L$$

$$P : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ 'no switch at this point' } \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ 'switch rows 2 and 4' } \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ 'no switches at this stage' } = P$$

Here is a verification:

$$LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & -1 & 4 & -3 \\ -1 & 1 & -3 & 0 \end{pmatrix}$$

$$PA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & 1 \\ -1 & 1 & -3 & 0 \\ 1 & -1 & 4 & -3 \\ 1 & 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & -1 & 4 & -3 \\ -1 & 1 & -3 & 0 \end{pmatrix}$$

$$\text{We are given that } b = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} \text{ so we compute } \hat{b} = Pb = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 1 \end{pmatrix}$$

Now we need to solve  $Lc = \hat{b}$  for  $c$ . This system is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 + c_2 \\ c_1 + c_3 \\ -c_1 - \frac{1}{2}c_3 + c_4 \end{pmatrix} = \hat{b} = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 1 \end{pmatrix} \text{ Thus } c_1 = 0, c_2 = 4, c_3 = 2,$$

and  $c_4 = 1 + 0 + \frac{1}{2}(2) = 2$ . Finally we need to solve  $Ux = c$  for  $x$ , that is

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x - y + 2z + w \\ 3y - 3z \\ 2z - 4w \\ -w \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 2 \end{pmatrix}. \text{ By back substitution, } w = -2.$$

Next  $2z - 4w = 2$ , so  $z = \frac{1}{2}(2 + 4(-2)) = -3$ . Then  $y = \frac{1}{3}(4 + 3(-3)) = -\frac{5}{3}$ . Finally  $x = -\frac{5}{3} - 2(-3) - (-2) = \frac{19}{3}$ . Here is a final check:

$$Ax = \begin{pmatrix} 1 & -1 & 2 & 1 \\ -1 & 1 & -3 & 0 \\ 1 & -1 & 4 & -3 \\ 1 & 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{19}{3} \\ -\frac{5}{3} \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

Problem 1.5.4

$$LL^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$L^{-1}L = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ so } L^{-1} \text{ is the inverse of } L.$$

$$\text{Next } MN = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & -c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -ca & 0 & 1 \end{pmatrix} \text{ so we don't have an inverse}$$

here. Let us notice that

$$M = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}. \text{ Therefore}$$

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ca-b & -c & 1 \end{pmatrix}. \text{ Here is a check:}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ca-b & -c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ca-b & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 1.5.5

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 0 \\ i & j & k & l \end{pmatrix} \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{pmatrix} =$$

$$\begin{pmatrix} aA+bE+cI+dM & aB+bF+cJ+dN & aC+bG+cK+dO & aD+bH+cL+dP \\ eA+fE+gI+hM & eB+fF+gJ+hN & eC+fG+gK+hO & eD+fH+gL+hP \\ 0 & 0 & 0 & 0 \\ iA+jE+kI+lM & iB+jF+kJ+lN & iC+jG+kK+lO & iD+jH+kL+lP \end{pmatrix}$$

Clearly in any product the zero row will produce another zero row as in the example. Thus the product can never be the identity matrix.

Problem 1.5.16

If any one of the diagonal entries is zero, then the matrix has a zero row and so cannot have an inverse (Problem 1.5.5). Next we just need to exhibit an inverse that works. Use the one given in the problem and multiply it out.

Problem 1.5.32 (d)

$$A: \begin{pmatrix} 1 & 1 & 5 \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 5 \\ 0 & 0 & -7 \\ 0 & -3 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 5 \\ 0 & -3 & -7 \\ 0 & 0 & -7 \end{pmatrix} = U$$

$$L: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = L$$

$$P: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = P$$

Now write  $U$  as

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & \frac{7}{3} \\ 0 & 0 & 1 \end{pmatrix} \text{ Thus the } LDV \text{ factorization, with verification, is}$$

$$PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 5 \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & -1 & 3 \\ 1 & 1 & -2 \end{pmatrix}$$

$$PA = LDV = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & \frac{7}{3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & -1 & 3 \\ 1 & 1 & -2 \end{pmatrix}$$