Math 415 - Assignment 10 Solutions

Problems: 8.2.1 (d), 8.2.1 (e), 8.2.1(k), 8.2.2, 8.2.6, 8.2.10, 8.2.14, 8.2.17, 8.2.20, 8.3.2 (h), 8.3.15 (e)

Problem 8.2.1 (d)

$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, so $\det \begin{pmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{pmatrix} = 3 - 2\lambda + \lambda^2 = (\lambda-1)^2 + 2 = 0 \Rightarrow \lambda = 1 \pm i\sqrt{2}$

Case 1 :$\lambda = 1 + i\sqrt{2} \Rightarrow (A - (1+i\sqrt{2})I)v = \begin{pmatrix} -i\sqrt{2} & 2 \\ -1 & -i\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -i\sqrt{2}a + 2b = 0, \Rightarrow b = i\frac{a}{\sqrt{2}}$ so choose $v = \begin{pmatrix} 1 \\ \frac{i}{\sqrt{2}} \end{pmatrix}$. We conclude that also $\lambda = 1 - i\sqrt{2}$ with eigenvector $v = \begin{pmatrix} 1 \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$

Problem 8.2.1 (e)

$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$, so $\det \begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix} = 12 - 19\lambda + 8\lambda^2 - \lambda^3 = -(\lambda-1)(\lambda-3)(\lambda-4) = 0$

Case $\lambda = 1$, $\begin{pmatrix} 3-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 3-1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a-b \\ -a+b-c \\ -b+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow b = 2a, c = \frac{1}{2}b = a$ so $v = \begin{pmatrix} a \\ 2a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ so choose $v = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Case $\lambda = 3$, $\begin{pmatrix} 3-3 & -1 & 0 \\ -1 & 2-3 & -1 \\ 0 & -1 & 3-3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b \\ -a-b-c \\ -b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow b = 0, a = -c$ so $v = \begin{pmatrix} -c \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ so choose $v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Case $\lambda = 4$, $\begin{pmatrix} 3-4 & -1 & 0 \\ -1 & 2-4 & -1 \\ 0 & -1 & 3-4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a-b \\ -a-2b-c \\ -b-c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a = -b, c = -b$ so $v = \begin{pmatrix} -b \\ b \\ -b \end{pmatrix} = b \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ so choose $v = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

Problem 8.2.1 (k)

$A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix}$,so $\det \begin{pmatrix} 4-\lambda & 0 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ -1 & 1 & 2-\lambda & 0 \\ 1 & -1 & 1 & 1-\lambda \end{pmatrix} = (4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda) = 0$

Case $\lambda = 4$, $\begin{pmatrix} 4-4 & 0 & 0 & 0 \\ 1 & 3-4 & 0 & 0 \\ -1 & 1 & 2-4 & 0 \\ 1 & -1 & 1 & 1-4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ a-b \\ -a+b-2c \\ a-b+c-3d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a = b, c = \frac{1}{2}(0) = 0, d = \frac{1}{3}(c) = 0$ so $v = \begin{pmatrix} b \\ b \\ 0 \\ 0 \end{pmatrix} = b \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ so choose $v = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Case $\lambda = 3$, $\begin{pmatrix} 4-3 & 0 & 0 & 0 \\ 1 & 3-3 & 0 & 0 \\ -1 & 1 & 2-3 & 0 \\ 1 & -1 & 1 & 1-3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ a \\ -a+b-c \\ a-b+c-2d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a =$ $0, c = b, d = \frac{1}{2}(0-b+b) = 0$ so $v = \begin{pmatrix} 0 \\ b \\ b \\ 0 \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ so choose $v = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

Case $\lambda = 2$, $\begin{pmatrix} 4-2 & 0 & 0 & 0 \\ 1 & 3-2 & 0 & 0 \\ -1 & 1 & 2-2 & 0 \\ 1 & -1 & 1 & 1-2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2a \\ a+b \\ -a+b \\ a-b+c-d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a = 0, b =$ $-a = 0, d = c$ so $v = \begin{pmatrix} 0 \\ 0 \\ c \\ c \end{pmatrix} = c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ so choose $v = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Case $\lambda = 1$, $\begin{pmatrix} 4-1 & 0 & 0 & 0 \\ 1 & 3-1 & 0 & 0 \\ -1 & 1 & 2-1 & 0 \\ 1 & -1 & 1 & 1-1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3a \\ a+2b \\ -a+b+c \\ a-b+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a = 0, b =$ $-\frac{1}{2}(a) = 0, c = a-b = 0$ so $v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix} = d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ so choose $v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Problem 8.2.2

(a) $\det \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} = \cos^2\theta - 2(\cos\theta)\lambda + \lambda^2 + \sin^2\theta = (\lambda - \cos\theta)^2 + \sin^2\theta = 0$, so $\lambda = \cos\theta \pm i\sin\theta$

(b) The eigenvalues are real when $\sin\theta = 0$, i.e. $\theta = 0, \pi, 2\pi$, etc.

(c) So for $0 < \theta < \pi$ R_θ does not have a real eigenvalue, and so $R_\theta - aI$ has non-zero determinant and so has an inverse.

Problem 8.2.6

$A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix} \Rightarrow$

$\lambda = 0, v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\lambda = i\sqrt{a^2+b^2+c^2}, v = \begin{pmatrix} -ca + ib\sqrt{a^2+b^2+c^2} \\ -bc - ia\sqrt{a^2+b^2+c^2} \\ a^2+b^2 \end{pmatrix}$

$\lambda = -i\sqrt{a^2+b^2+c^2}, v = \begin{pmatrix} -ca - ib\sqrt{a^2+b^2+c^2} \\ -bc + ia\sqrt{a^2+b^2+c^2} \\ a^2+b^2 \end{pmatrix}$

Problem 8.2.10

(a) $A(cv) = cAv = c(\lambda v) = \lambda(cv)$, so cv is also an eignevector.

(b) Suppose that $Av = \lambda v$ and $Aw = \lambda w$. Then $A(av+bw) = aAv + bAw = a\lambda v + b\lambda w = \lambda(av+bw)$ and so $av + bw$ is also an eigenvector for any scalars a and b.

(c) If $Av = \lambda v$ and $Aw = \mu w$ with $\lambda \neq \mu$, then $cv+dw$ is an eigenvector too if there is ab=n eigenvalue β such that $A(cv+dw) = \beta(cv+dw)$. This equation implies that $\beta(cv+dw) = cAv+dAw = c\lambda v + d\mu w$ and so $c(\beta-\lambda)v + d(\beta-\mu)w = 0$. Since v and w correspond to different eigenvalues, they are linearly independent and so this linear combination is zero only if $c(\beta-\lambda)=0$ and $d(\beta-\mu)=0$. But these can't be true since $c\neq 0, d\neq 0$ and $\lambda\neq\mu$.

Problem 8.2.14

(a) $A=\begin{pmatrix}1&4&4\\3&-1&0\\0&2&3\end{pmatrix}\Rightarrow\lambda=-3$ with $v=\begin{pmatrix}2\\-3\\1\end{pmatrix}$, $\lambda=1$ with $v=\begin{pmatrix}2\\3\\-3\end{pmatrix}$, $\lambda=5$ with $v=\begin{pmatrix}2\\1\\1\end{pmatrix}$

(b) $\text{tr}A=1-1+3=3=-3+1+5$

(c) $\det A=\det\begin{pmatrix}1&4&4\\3&-1&0\\0&2&3\end{pmatrix}=-15=(-3)(1)(5)$

Problem 8.2.17

Just do this with an example:

If $A=\begin{pmatrix}a&b&c&d\\0&e&f&g\\0&0&h&k\\0&0&0&m\end{pmatrix}$ then $\det(A-\lambda I)=\det\begin{pmatrix}a-\lambda&b&c&d\\0&e-\lambda&f&g\\0&0&h-\lambda&k\\0&0&0&m-\lambda\end{pmatrix}$

$=(a-\lambda)(e-\lambda)(h-\lambda)(m-\lambda)=0$ so the diagonal entries are the eigenvalues

Problem 8.2.20

If $Av=\lambda v$, then $A^2v=A(Av)=A(\lambda v)=\lambda Av=\lambda(\lambda v)=\lambda^2 v$. "That was easy!"

Problem 8.3.2 (h)

$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&1&-1&0\\1&0&-1&0\end{pmatrix}\Rightarrow$ eigenvalues are along the diagonal (lower triangular), so $\lambda=1,1,-1,0$

$\lambda=0\Rightarrow(A-0I)v=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&1&-1&0\\1&0&-1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a\\b\\-a+b-c\\a-c\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

We conclude that $a=b=c=0$ and d is arbitrary, so $V_0=\text{span}\left\{\begin{pmatrix}0&0&0&1\end{pmatrix}^T\right\}$

$\lambda=-1\Rightarrow(A+I)v=\begin{pmatrix}2&0&0&0\\0&2&0&0\\-1&1&0&0\\1&0&-1&1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}2a\\2b\\-a+b\\a-c+d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

We conclude that $a=b=0$ and $c=d$ is arbitrary, so $V_{-1}=\text{span}\left\{\begin{pmatrix}0&0&1&1\end{pmatrix}^T\right\}$

$\lambda=1\Rightarrow(A-I)v=\begin{pmatrix}0&0&0&0\\0&0&0&0\\-1&1&-2&0\\1&0&-1&-1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}0\\0\\-a+b-2c\\a-c-d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

We conclude that $b=a+2c, d=a-c$, so

$\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a\\a+2c\\c\\a-c\end{pmatrix}=a\begin{pmatrix}1\\1\\0\\1\end{pmatrix}+c\begin{pmatrix}0\\2\\1\\-1\end{pmatrix}$

Therefore $V_1=\text{span}\left\{\begin{pmatrix}1&1&0&1\end{pmatrix}^T,\begin{pmatrix}0&2&1&-1\end{pmatrix}^T\right\}$

Problem 8.3.15 (e)

$$A = \begin{pmatrix} 8 & 0 & -3 \\ -3 & 0 & -1 \\ 3 & 0 & -2 \end{pmatrix}, \text{ eigenvectors: } \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \leftrightarrow 0, \left\{ \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} \right\} \leftrightarrow -1, \left\{ \begin{pmatrix} 21 \\ -10 \\ 7 \end{pmatrix} \right\} \leftrightarrow 7$$

$$\text{so } S = \begin{pmatrix} 0 & 1 & 21 \\ 1 & 6 & -10 \\ 0 & 3 & 7 \end{pmatrix}, \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$\text{Check: } S^{-1}AS = \begin{pmatrix} 0 & 1 & 21 \\ 1 & 6 & -10 \\ 0 & 3 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 8 & 0 & -3 \\ -3 & 0 & -1 \\ 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 21 \\ 1 & 6 & -10 \\ 0 & 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{7} & 1 & -\frac{17}{7} \\ -\frac{1}{8} & 0 & \frac{3}{8} \\ \frac{3}{56} & 0 & -\frac{1}{56} \end{pmatrix} \begin{pmatrix} 8 & 0 & -3 \\ -3 & 0 & -1 \\ 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 21 \\ 1 & 6 & -10 \\ 0 & 3 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$