Math 415 - Assignment 11 Solutions

Problems: 8.4.1 (d), 8.4.2, 9.1.12 (e), 9.1.18, 9.2.1

Problem 8.4.1 (d)

Set $A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix}$. Then $0 = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 4 \\ 0 & 1-\lambda & 3 \\ 4 & 3 & 1-\lambda \end{pmatrix}$

$= -24 + 22\lambda + 3\lambda^2 - \lambda^3 = -(\lambda - 1)(\lambda - 6)(\lambda + 4)$ so $\lambda = 1, \lambda = 6, \lambda = -4$

Case $\lambda = 1$: $(A - I)v = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 3 \\ 4 & 3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4c \\ 3c \\ 4a + 3b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$c = 0, 3b = -4a$ and so v is a multiple of $\begin{pmatrix} 3 & -4 & 0 \end{pmatrix}^T$. The corresponding unit vector is $v = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \end{pmatrix}^T$

Case $\lambda = 6$: $(A - 6I)v = \begin{pmatrix} -5 & 0 & 4 \\ 0 & -5 & 3 \\ 4 & 3 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -5a + 4c \\ -5b + 3c \\ 4a + 3b - 5c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$a = \frac{4}{5}c, b = \frac{3}{5}c$ and so v is a multiple of $\begin{pmatrix} \frac{4}{5} & \frac{3}{5} & 1 \end{pmatrix}^T$. The corresponding unit vector is $v = \begin{pmatrix} \frac{4}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T$

Case $\lambda = -4$: $(A + 4I)v = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 3 \\ 4 & 3 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5a + 4c \\ 5b + 3c \\ 4a + 3b + 5c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$a = -\frac{4}{5}c, b = -\frac{3}{5}c$ and so v is a multiple of $\begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} & 1 \end{pmatrix}^T$. The corresponding unit vector is $v = \begin{pmatrix} -\frac{4}{5\sqrt{2}} & -\frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T$

Problem 8.4.2

(a) $A = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} \Rightarrow$ eigenvalues: $\frac{5}{2} + \frac{1}{2}\sqrt{17}, \frac{5}{2} - \frac{1}{2}\sqrt{17} \Rightarrow$ positive definite

(b) $A = \begin{pmatrix} -2 & 2 \\ 2 & 6 \end{pmatrix} \Rightarrow$ eigenvalues: $2\sqrt{5} + 2, 2 - 2\sqrt{5} \Rightarrow$ not definite

(c) $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow$ eigenvalues: $0, 1, 3 \Rightarrow$ positive semi-definite

(d) $A = \begin{pmatrix} 4 & -1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & 4 \end{pmatrix} \Rightarrow$ eigenvalues: $6, 3 + \sqrt{3}, 3 - \sqrt{3} \Rightarrow$ positive definite

Problem 9.1.12 (e)

Here we should set $A = \begin{pmatrix} 3 & -8 & 2 \\ -1 & 2 & 2 \\ 1 & -4 & 2 \end{pmatrix}$ and do an eigenvalue analysis:

$0 = \det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -8 & 2 \\ -1 & 2-\lambda & 2 \\ 1 & -4 & 2-\lambda \end{pmatrix}$

$= 8 - 14\lambda + 7\lambda^2 - \lambda^3 = -(\lambda - 1)(\lambda - 2)(\lambda - 4)$ so $\lambda = 1, \lambda = 2, \lambda = 4$

Case $\lambda = 1$: $(A - I)v = \begin{pmatrix} 2 & -8 & 2 \\ -1 & 1 & 2 \\ 1 & -4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a - 8b + 2c \\ -a + b + 2c \\ a - 4b + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

(reversing the order of the equations) $\begin{pmatrix} 1 & -4 & 1 \\ -1 & 1 & 2 \\ 2 & -8 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -4 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$

$b = c, a = 4b - c = 3c$ and so choose v to be $\begin{pmatrix} 3 & 1 & 1 \end{pmatrix}^T$

Case $\lambda = 2$: $(A - 2I)v = \begin{pmatrix} 1 & -8 & 2 \\ -1 & 0 & 2 \\ 1 & -4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - 8b + 2c \\ -a + 2c \\ a - 4b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$b = \frac{1}{4}a, c = \frac{1}{2}a$ and so by setting $a = 4$ we can let v be $\begin{pmatrix} 4 & 1 & 2 \end{pmatrix}^T$

Case $\lambda = 4$: $(A - 4I)v = \begin{pmatrix} -1 & -8 & 2 \\ -1 & -2 & 2 \\ 1 & -4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a - 8b + 2c \\ -a - 2b + 2c \\ a - 4b - 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

(reversing the order of the equations) $\begin{pmatrix} 1 & -4 & -2 \\ -1 & -2 & 2 \\ -1 & -8 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -4 & -2 \\ 0 & -6 & 0 \\ 0 & -12 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -4 & -2 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$

$b = 0, a = 2c$ and so we can set $v = \begin{pmatrix} 2 & 0 & 1 \end{pmatrix}^T$

Therefore the general solution is

$$u(t) = c_1 e^t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3c_1 e^t + 4c_2 e^{2t} + 2c_3 e^{4t} \\ c_1 e^t + c_2 e^{2t} \\ c_1 e^t + 2c_2 e^{2t} + c_3 e^{4t} \end{pmatrix}$$

Problem 9.1.18

(a) Set $K = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ and do an eigenvalue analysis:

$0 = \det(K - \lambda I) = \det \begin{pmatrix} 1 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{pmatrix}$

$= -3\lambda + 4\lambda^2 - \lambda^3 = -\lambda (-1 + \lambda)(\lambda - 3)$ so $\lambda = 0, \lambda = 1, \lambda = 3$

Case $\lambda = 0$: $(K - 0I)v = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - b \\ -a + 2b - c \\ -b + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$b = c, a = b = c$ and so choose v to be $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$

Case $\lambda = 1$: $(K - I)v = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b \\ -a + b - c \\ -b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$b = 0, a = -c$ and so we can let v be $\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T$

Case $\lambda = 3$: $(K - 3I)v = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a - b \\ -a - b - c \\ -b - 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$a = -\frac{1}{2}b, c = -\frac{1}{2}b$ and so we can set $v = \begin{pmatrix} -1 & 2 & -1 \end{pmatrix}^T$

(b) Orthogonality is clear by inspection. And since all eigenvalues are positive, K is positive definite.

(c) The general solution is

$$u(t) = c_1 e^{0t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 e^t - c_3 e^{3t} \\ c_1 + 2c_3 e^{3t} \\ c_1 + c_2 e^t - c_3 e^{3t} \end{pmatrix}$$

so we need to solve $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = u(0) = \begin{pmatrix} c_1 - c_2 - c_3 \\ c_1 + 2c_3 \\ c_1 + c_2 - c_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow$

$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & -1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 0 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -6 & -4 \end{pmatrix} \Rightarrow$

$c_3 = \frac{2}{3}, c_2 = 1 - 3c_3 = -1, c_1 = 1 + c_2 + c_3 = \frac{2}{3}$. Hence $u(t) = \begin{pmatrix} \frac{2}{3} + e^t - \frac{2}{3}e^{3t} \\ \frac{2}{3} + \frac{4}{3}e^{3t} \\ \frac{2}{3} - e^t - \frac{2}{3}e^{3t} \end{pmatrix}$

Problem 9.2.1

(a) Asymptotically stable: $A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}$, eigenvalues: $-2+i, -2-i$ (real part is negative)

(b) Unstable: $A = \begin{pmatrix} 2 & -5 \\ 1 & -1 \end{pmatrix}$, eigenvalues: $\frac{1}{2}+\frac{1}{2}i\sqrt{11}, \frac{1}{2}-\frac{1}{2}i\sqrt{11}$ (real part is positive)

(c) Asymptotically stable: $A = \begin{pmatrix} -1 & -2 \\ 2 & -5 \end{pmatrix}$, eigenvalues: $-3, -3$ (eigenvalues are negative)

(d) Stable: $A = \begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix}$, eigenvalues: $4i, -4i$ (real parts are zero)

(e) Stable: $A = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & 1 \\ -3 & -3 & 2 \end{pmatrix}$, eigenvalues: $0, -1, -1$ (there is a 0 eigenvalue)

(f) Unstable: $A = \begin{pmatrix} -1 & -2 & 0 \\ 6 & 3 & -4 \\ 4 & 0 & -3 \end{pmatrix}$, eigenvalues: $1, -1+2i, -1-2i$ (one eigenvalue is positive)

(g) Asymptotically stable: $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 1 \\ -4 & 1 & -5 \end{pmatrix}$, eigenvalues: $-2, -1, -1$ (all negative eigenvalues)

(h) Stable: $A = \begin{pmatrix} 1 & 1 & -1 \\ -2 & -3 & 3 \\ 0 & -1 & 1 \end{pmatrix}$, eigenvalues: $0, 0, -1$ (there is a 0 eigenvalue)