

Math 415 - Assignment 2 Solutions

Problems: 1.6.2, 1.6.7, 1.6.9, 1.6.17, 1.6.21, 1.6.25 (d), 1.8.2 (d), 1.8.2 (e), 1.8.5, 1.8.7 (g), 1.8.7 (h) 1.8.8 (g), 1.8.8 (h), 1.8.27 (b), 1.9.1 (f), 1.9.6

Problem 1.6.2

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 2 & 0 \\ -3 & 4 \end{pmatrix}, A^T = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & 1 \\ -1 & 2 \\ -1 & 1 \end{pmatrix}$$

$$B^T = \begin{pmatrix} -1 & 2 \\ 2 & 0 \\ -3 & 4 \end{pmatrix}^T = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 4 \end{pmatrix}. \text{ Therefore}$$

$$(AB)^T = B^T A^T = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & 6 \end{pmatrix} \text{ and}$$

$$(BA)^T = A^T B^T = \begin{pmatrix} 3 & 1 \\ -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 6 & -5 \\ 5 & -2 & 11 \\ 3 & -2 & 7 \end{pmatrix}$$

Problem 1.6.7

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \text{ Then } AA^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}. \text{ The (1,1) positions are equal only if either } b = c \text{ or if } b = -c. \text{ In either case the (2,2) positions are equal. If } b = c, \text{ then the (1,2) and (2,1) positions are also equal. If } b = -c, \text{ then the (1,2) and (2,1) positions are equal if and only if } ac + bd = ac - cd = -ac + cd = ab + cd, \text{ that is, if } a = d. \text{ Thus the matrices of interest have the form } \begin{pmatrix} a & b \\ b & d \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Problem 1.6.9

We simply need to find matrices B and C such that $B(AA^T) = (AA^T)B = I$ and $C(A^T A) = (A^T A)C = I$. But $B = A^{-T} A^{-1}$ and $C = A^{-1} A^{-T}$ do the job. Both exist since A has an inverse.

Problem 1.6.17

(a) In this case we need $2a - 1 = a$, and so $a = 1$

(b) For this case we need $a = -1, b = 2$ and $c = 3$

(c) Here we need $a + 2b - 2c = 6, -a + b + c = -4, b - c = 4$

$$\begin{pmatrix} 1 & 2 & -2 & 6 \\ -1 & 1 & 1 & -4 \\ 0 & 1 & -1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -2 & 6 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -2 & 6 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -\frac{2}{3} & \frac{10}{3} \end{pmatrix}, \Rightarrow c = -5, b = -1, a = -2$$

Problem 1.6.21

$(AB)^T = B^T A^T = BA \neq AB$. False!

Problem 1.6.25 (d)

$$\begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -1 & 0 \\ 0 & 3 & 0 & -8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & -6 & -17 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -\frac{49}{5} \end{pmatrix}$$

$$\text{Therefore } LDL^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 3 & \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -\frac{49}{5} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & \frac{6}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 1.8.2 (d)
 $\left(\begin{array}{ccc|c} 2 & -6 & 4 & 2 \\ -1 & 3 & -2 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & -6 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$, so the system is compatible and we treat x_2 and x_3 as free variables:

$$x_1 = \frac{1}{2}(2 + 6x_2 - 4x_3) = 1 + 3x_2 - 2x_3$$

$$\text{Check: } \left(\begin{array}{ccc} 2 & -6 & 4 \\ -1 & 3 & -2 \end{array} \right) \begin{pmatrix} 1 + 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Problem 1.8.2 (e)
 $\left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 4 & 5 & 7 & 15 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 13 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 10 \end{array} \right)$, so the system is compatible

and has a unique solution.

$$x_3 = -10, x_2 = 3 - 2x_3 = 23, x_1 = \frac{1}{2}(1 - 2(23) - 3(-10)) = -\frac{15}{2}$$

$$\text{Check: } \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 5 & 7 \end{pmatrix} \begin{pmatrix} -\frac{15}{2} \\ 23 \\ -10 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 15 \end{pmatrix}$$

Problem 1.8.5
 $A = \begin{pmatrix} 1 & 1 & b & 1 \\ b & 3 & -1 & -2 \\ 3 & 4 & 1 & c \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & b & 1 \\ 0 & 3-b & -1-b^2 & -2-b \\ 0 & 1 & 1-3b & c-3 \end{pmatrix}$

Case 1: $b = 3$ In this case $A \rightarrow \begin{pmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & -10 & -5 \\ 0 & 1 & -8 & c-3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & -8 & c-3 \\ 0 & 0 & -10 & -5 \end{pmatrix}$ which has a

unique solution

Case 2: $b \neq 3$. In this case interchange rows 2 and 3 and continue the elimination:

$$A \rightarrow \begin{pmatrix} 1 & 1 & b & 1 \\ 0 & 1 & 1-3b & c-3 \\ 0 & 3-b & -1-b^2 & -2-b \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & b & 1 \\ 0 & 1 & 1-3b & c-3 \\ 0 & 0 & -4+10b-4b^2 & 7-4b-(3-b)c \end{pmatrix}$$

We will have no solution if $-4 + 10b - 4b^2 = 0$ (i.e. $b = 2$ or $b = \frac{1}{2}$) and $7 - 4b - (3 - b)c \neq 0$, i.e. $b = 2$ and $c \neq -1$, or $b = \frac{1}{2}$ and $c \neq 2$. If either $b = 2$ and $c = -1$, or $b = \frac{1}{2}$ and $c = 2$ then we have a zero row. This means there is a free variable and thus infinitely many solutions. If b is neither 2 nor $\frac{1}{2}$, then we have 3 pivots and hence a unique solution.

Problems 1.8.7 (g) and 1.8.8 (g)
 $A = \begin{pmatrix} 0 & -3 \\ 4 & -1 \\ 1 & 2 \\ -1 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 4 & -1 \\ 0 & -3 \\ -1 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -9 \\ 0 & -3 \\ 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -9 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = U$
 $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 0 \\ -1 & \frac{1}{3} & 0 & 1 \end{pmatrix} = L$
 $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = P$

We conclude that $\text{rank}(A) = 2$, i.e. 2 pivots. And here is a check that $PA = LU$:

$$PA = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -3 \\ 4 & -1 \\ 1 & 2 \\ -1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & -1 \\ 0 & -3 \\ -1 & -5 \end{pmatrix}$$

$$LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 0 \\ -1 & \frac{1}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -9 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & -1 \\ 0 & -3 \\ -1 & -5 \end{pmatrix}$$

Problems 1.8.7 (h) and 1.8.8 (h)

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -3 & -1 \\ 4 & -1 & 3 & 2 \\ 0 & 3 & -5 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & -2 \\ 0 & 3 & -5 & -2 \\ 0 & 3 & -5 & -2 \\ 0 & 3 & -5 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} = L$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \text{ We conclude that } \text{rank}(A) = 2, \text{ i.e. } 2 \text{ pivots. And here is a check that}$$

$PA = LU$:

$$PA = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -3 & -1 \\ 4 & -1 & 3 & 2 \\ 0 & 3 & -5 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -3 & -1 \\ 4 & -1 & 3 & 2 \\ 0 & 3 & -5 & -2 \end{pmatrix}$$

$$LU = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -3 & -1 \\ 4 & -1 & 3 & 2 \\ 0 & 3 & -5 & -2 \end{pmatrix}$$

Problem 1.8.27 (b)

$$A = \begin{pmatrix} 1 & k & 4 \\ k & 1 & 2 \\ 2 & k & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & k & 4 \\ 0 & 1 - k^2 & 2 - 4k \\ 0 & -k & 0 \end{pmatrix}. \text{ If } k = 0, \text{ then the last row is zero and so there will}$$

be a non-trivial solution. If $k \neq 0$, then interchange rows 2 and 3 and reduce further:

$$A \rightarrow \begin{pmatrix} 1 & k & 4 \\ 0 & -k & 0 \\ 0 & 1 - k^2 & 2 - 4k \end{pmatrix} \rightarrow \begin{pmatrix} 1 & k & 4 \\ 0 & -k & 0 \\ 0 & 0 & 2 - 4k \end{pmatrix}. \text{ This shows that also when } k = \frac{1}{2} \text{ we will}$$

have non-trivial solutions

Problem 1.9.1 (f)

$$A = \begin{pmatrix} 1 & -2 & 1 & 4 \\ 2 & -4 & 0 & 0 \\ 3 & -4 & 2 & 5 \\ 0 & 2 & -4 & -9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & -2 & -8 \\ 0 & 2 & -1 & -7 \\ 0 & 2 & -4 & -9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & 4 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & -2 & -8 \\ 0 & 2 & -4 & -9 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & 1 & 4 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & 4 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 10 \end{pmatrix} = U$$

So $-\det A = 1(2)(-2)(10)$ or $\det A = 40$

Problem 1.9.6

$$A = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix} \rightarrow \begin{pmatrix} b & -c & 0 \\ -a & 0 & c \\ 0 & a & -b \end{pmatrix} \rightarrow \begin{pmatrix} b & -c & 0 \\ 0 & -\frac{ac}{b} & c \\ 0 & a & -b \end{pmatrix}$$

$\rightarrow \begin{pmatrix} b & -c & 0 \\ 0 & -\frac{ac}{b} & c \\ 0 & 0 & 0 \end{pmatrix} = U$. This shows that $\det A = b(-\frac{ac}{b})0 = 0$ and so A is NEVER invertible.