Math 415 - Assignment 3 Solutions

Problems: 2.1.7, 2.2.1, 2.2.7, 2.2.9, 2.3.2, 2.3.6, 2.3.7, 2.3.8, 2.3.22, 2.3.23, 2.3.25

Problem 2.1.7

(a) Not (i) since it is a scalar, not (v) since it is in M_{2x2}, not (vi) since it is in R^3.

(b) $-5\left(\begin{pmatrix} x-y \\ xy \end{pmatrix} + \begin{pmatrix} e^x \\ \cos y \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} -5x+5y-5e^x-5 \\ -5xy-5\cos y-15 \end{pmatrix}$

(c) The zero element is the zero function $\mathbf{0}(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Problem 2.2.1

(a) Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $w = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ both satisfy the equation, i.e. $x-y+4z=0, X-Y+4Z=0$.

For scalars a and b we have $av+bw = \begin{pmatrix} ax+bX \\ ay+bY \\ az+bZ \end{pmatrix}$. We need to show that this vector also satisfies the equation. Here it is: $(ax+bX)-(ay+bY)+4(az+bZ) = a(x-y+4z)+b(X-Y+4Z) = a \times 0 + b \times 0 = 0$.

(b) the zero vector must lie in a subspace, but $x=0, y=0, z=0$ does not satisfy $x-y+4z=1$.

Problem 2.2.7

(a) Yes: $a\begin{pmatrix} x \\ ... \\ x \end{pmatrix} + b\begin{pmatrix} y \\ ... \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ ... \\ ax+by \end{pmatrix}$ which again has all equal entries

(b) No: $-2\begin{pmatrix} 1 \\ ... \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ ... \\ -2 \end{pmatrix}$ which has negative entries

(c) Yes: $a\begin{pmatrix} 0 \\ ...\text{ (anything)} \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ ...\text{ (anything)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ ...\text{ (anything)} \\ 0 \end{pmatrix}$ which again has first and last entries 0

(d) Yes: proof is similar to 2.2.1 above

(e) No: the zero vector does not satisfy this condition. Alternately, if x satisfies it, then the scalar multiple $-x$ does not.

Problem 2.2.9

Convince your self in the case of general 3x3 or 4x4 matrices:

$a\begin{pmatrix} 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ d & e & 0 & 0 \\ f & g & h & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ D & E & 0 & 0 \\ F & G & H & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ ac+bC & 0 & 0 & 0 \\ ad+bD & ae+bE & 0 & 0 \\ af+bF & ag+bG & ah+bH & 0 \end{pmatrix}$, again strictly lower triangular.

Problem 2.3.2

We need to find real numbers c_1, c_2, c_3 such that

$$\begin{pmatrix} -3 \\ 7 \\ 6 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -3 \\ -2 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 6 \\ 3 \\ 4 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ 4 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 \\ -3 & 6 & 4 \\ -2 & 3 & 6 \\ 0 & 4 & -7 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

so Gaussian eliminate the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ -3 & 6 & 4 & 7 \\ -2 & 3 & 6 & 6 \\ 0 & 4 & -7 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & 0 & -2 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 4 & -7 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow c_3 = 1, c_2 = 2c_3 = 2, c_1 = -3 + 2c_2 + 2c_3 = 3$$

Check: $3\begin{pmatrix} 1 \\ -3 \\ -2 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 6 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \\ 6 \\ 1 \end{pmatrix}$

Problem 2.3.6

V will be a subspace of U if $\mathbf{v}_1$ and $\mathbf{v}_2$ are both in U (since U is a vector space itself). So we need to find constants c_1, c_2 and d_1, d_2 such that $\mathbf{v}_1 = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = d_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + d_2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$. Let's solve these together by Gaussian reducing a double augmented matrix: $\left(\begin{array}{cc|cc} 1 & 2 & 5 & 3 \\ 2 & -1 & 0 & 1 \\ 3 & 0 & 3 & 3 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 5 & 3 \\ 0 & -5 & -10 & -5 \\ 0 & -6 & -12 & -6 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 5 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Since there are two pivots, we can uniquely solve for $c_1, ..., d_2$ (we don't need to solve explicitly. It is enough to know that we can to say that $\mathbf{v}_1$ and $\mathbf{v}_2$ are in U). The next question is whether U and V are the same. It is enough then to show that U is a subspace of V. We can do this in two ways. One is to repeat the above calculation with U and V interchanged. This leads to the calculation $\left(\begin{array}{cc|cc} 5 & 3 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 3 & 3 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 5 & 3 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & \frac{6}{5} & \frac{12}{5} & -\frac{6}{5} \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 5 & 3 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ and again we have two pivots. Thus $\mathbf{u}_1$ and $\mathbf{u}_2$ are in V and so $U = V$. A second approach is simply to show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. The calculation just above shows that the matrix with $\mathbf{v}_1$ and $\mathbf{v}_2$ as columns has two pivots and this demonstrates the linear independence.

Problem 2.3.7

(a) $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Done!

(b) Since $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} +$ $d\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the six special matrices shown here span the set of 3x3 symmetric matrices.

Problem 2.3.8

(a) Yes: Since we are given 3 polynomials of degree 2, if they are linearly independent, then they will span $P^{(2)}$ (since this space has dimension 3). Alternately, we need to show that a general polynomial of degree 2 can be written as a linear combo of these three. This leads to the problem $p(x) = a_0 + a_1 x + a_2 x^2 = A(x^2+1) + B(x^2-1) + C(x^2+x+1) = (A-B+C) + Cx + (A+B+C)x^2$. We conclude that $A-B+C = a_0, C = a_1, A+B+C = a_2$. We solve these to get $A = \frac{1}{2}(a_0+a_2) - a_1, B = \frac{1}{2}(a_2 - a_0), C = a_1$. Check:.$\left(\frac{1}{2}(a_0+a_2) - a_1\right)(x^2+1) + \frac{1}{2}(a_2 - a_0)(x^2-1) + a_1(x^2+x+1) = a_0 + a_2 x^2 + a_1 x$

(b) Yes: Here we need to solve $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = A(x^3-1) + B(x^2+1) + C(x-1) + D = (D-A+B-C) + Cx + Bx^2 + Ax^3$. This is satified by taking $A = a_3, B = a_2, C = a_1, D = a_0 + a_3 - a_2 + a_1$, and so $P^{(3)}$ is spanned by

these 4 polynomials.
(c) No: Here we need to solve
$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = Ax^3 + B(x^2+1) + C(x^2-x) + D(x+1) = (D+B) + (D-C)x + (B+C)x^2 + Ax^3$. This means we need to solve $A = a_3, B+C = a_2, D-C = a_1, D+B = a_0$. Write these as the system

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} B \\ C \\ D \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix}$$ and use Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & a_2 \\ 0 & -1 & 1 & a_1 \\ 1 & 0 & 1 & a_0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a_2 \\ 0 & -1 & 1 & a_1 \\ 0 & -1 & 1 & a_0 - a_2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a_2 \\ 0 & -1 & 1 & a_1 \\ 0 & 0 & 0 & a_0 - a_2 - a_1 \end{array}\right)$$

Since this system is not compatible when $a_0 - a_2 - a_1 \neq 0$, we conclude that not all functions in $P^{(3)}$ can be written as a linear combo of these four polynomials.

Problem 2.3.22

(a) $$\begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$. Since there are 3 pivots, the vectors are linearly independent.

(b) Consider the augmented system with all four vectors included:

$$\left(\begin{array}{ccc|cccc} 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|cccc} 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 3 & -3 & 0 & -2 & 0 & 0 \\ 0 & 3 & -3 & 0 & -1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|cccc} 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|cccc} 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

The zero row in the first matrix has zeros in the second matrix for vectors (i), (iii) and (iv). Thus, each of these is in the span of the vectors in part (A). Vector (ii) leads to an imcompatible system and so is not in the span.
(c) This time do the elimination as follows:

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & a \\ 0 & 3 & -2 & b \\ 2 & -1 & 1 & c \\ 1 & 1 & -1 & d \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 2 & a \\ 0 & 3 & -2 & b \\ 0 & 3 & -3 & -2a+c \\ 0 & 3 & -3 & -a+d \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 2 & a \\ 0 & 3 & -2 & b \\ 0 & 0 & -1 & -b-2a+c \\ 0 & 0 & -1 & -b-a+d \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 2 & a \\ 0 & 3 & -2 & b \\ 0 & 0 & -1 & -b-2a+c \\ 0 & 0 & 0 & a-c+d \end{array}\right)$$. Now we see that we need $a - c + d = 0$ for $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ to be in the span.

Problems 2.3.23

(a) Let's do (a) and (c) together with the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & -2 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & 0 & -2 & \frac{1}{2} \end{pmatrix}$ Since the initial 4x4 matrix has 4 pivots, the vectors are linearly independent.

(b) Since R^4 has dimension 4, any set of 4 linearly independent vectors from R^4 has to span R^4.

(c) By part (a) if $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$ then c_1, c_2, c_3, c_4 must satisfy $-2c_4 = \frac{1}{2}, -2c_3 - 2c_4 = -1, -2c_2 - c_3 - c_4 = -1, c_1 + c_2 + c_3 + c_4 = 1$. These have solution $c_4 = -\frac{1}{4}, c_3 = -\frac{1}{2}(-1 + 2(-\frac{1}{4})) = \frac{3}{4}, c_2 = -\frac{1}{2}(-1 + \frac{3}{4} - \frac{1}{4}) = \frac{1}{4}, c_1 = 1 - \frac{1}{4} - \frac{3}{4} + \frac{1}{4} = \frac{1}{4}$.

Check:

$$\frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \frac{3}{4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Problems 2.3.25

False: Write $c_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$+c_5 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + c_6 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1 + c_6 & c_2 + c_4 & c_3 + c_5 \\ c_3 + c_4 & c_1 + c_5 & c_2 + c_6 \\ c_2 + c_5 & c_3 + c_6 & c_1 + c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

This leads to to nine equations $c_1 + c_6 = 0, c_3 + c_4 = 0, c_2 + c_5 = 0, c_2 + c_4 = 0, c_1 + c_5 = 0, c_3 + c_6 = 0, c_3 + c_5 = 0, c_2 + c_6 = 0, c_1 + c_4 = 0$. These can be written as the 9x6 system

$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and so we need to reduce this system. Here it is!

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are only 5 pivots, there is one free variable and hence many nontrivial solutions for the

c_i's. Here is an example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$