Math 415 - Assignment 4 Solutions

Problems: 2.4.2, 2.4.3 (c), 2.4.6, 2.4.8 (a0, 2.4.8 (b), 2.4.11, 2.4.15, 2.5.3, 2.5.4, 2.5.21 (c), 2.5.23, 2.6.2, 2.6.12

Problem 2.4.2

(a) No. R^3 is 3 dimensional and so any basis of it needs exactly 3 vectors

(b) Check linear independence: $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 3 & 3 \\ -5 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 3 \\ 0 & -1 & 1 \\ 0 & 15 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 30 \end{pmatrix}$

Since there are 3 pivots, these vectors are linearly independent and so form a basis of R^3.

(c) Check linear independence: $\begin{pmatrix} 0 & -1 & 1 \\ 4 & 0 & -8 \\ -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Here there are only two pivots, so the vectors are linearly dependent, and so not a basis.

(d) No: more than 3 vectors cannot form a basis for a space of dimension 3

Problem 2.4.3 (c)

Look at the hyperplane equation $x + 2y + z - w = 0$ as a single linear equation to be solved. We treat x as the basic variable and y, z, w as free and solve: $x = -2y - z + w$. Thus the most general vector in the hyperplane is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2y - z + w \\ y \\ z \\ w \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The three vectors on the right hand side are a basis for the hyperplane.

Problem 2.4.6

(a) as in the last problem, let us find a basis: solve $x - 2y - 4z = 0$ for x to give $x = 2y + 4z$ and then write the most general vector in the plane as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y + 4z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

We see that a basis consists of two vectors (i.e. the plane has dimension 2) and the basis here is one of the one's listed. To handle the other one, the two vectors involved must be inearly independent and their span must lie in the plane. For linear independence we check:

$$\begin{pmatrix} 2 & 0 \\ -1 & 2 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Since there are two pivots, we have independence. Now we just have to check the span. The span of these vectors consists of all vectors of the form

$$a \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2a \\ -a + 2b \\ a - b \end{pmatrix}$$

Since $(2a) - 2(-a + 2b) - 4(a - b) = 0$, this vector satisfies $x - 2y - 4z = 0$ for all a and b, i.e. for all vectors in the span.

(b) We need to find scalars c_1 and c_2 such that

$$c_1 \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

and scalars c_3 and c_4 such that

$$c_3 \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

We can do these two together through the following double elimination:

$$\left(\begin{array}{cc|cc} 4 & 2 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We conclude that $c_2 = -1, c_4 = 2, c_1 = \frac{1}{2} - \frac{1}{2}c_2 = 1, c_3 = 0 - \frac{1}{2}c_4 = -1$. Here is a check:

$$\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

(c) If v_1 and v_2 are the vectors in the first basis, then we can form new bases in many ways. For example, $2v_1$ and $3v_2$ constitute a new basis. So are v_1 and $v_1 - v_2$. Any two linear combinations of v_1 and v_2 that are linearly independent will make up a new basis.

Problem 2.4.8 (a)

First solve $Ax = 0$:

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 0 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -6 & 5 & -4 \end{pmatrix}$$

so $y = \frac{5}{6}z - \frac{2}{3}w$ and $x = -2(\frac{5}{6}z - \frac{2}{3}w) + z - w = -\frac{2}{3}z + \frac{1}{3}w$ and therefore the solutions are given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}z + \frac{1}{3}w \\ \frac{5}{6}z - \frac{2}{3}w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} -\frac{2}{3} \\ \frac{5}{6} \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$$

The latter two vectors form a basis for the subspace and so the subspace has dimension 2

Problem 2.4.8 (b)

We proceed as in the last problem. Since $p(x) = ax^2 + bx + c$, the condition $p(1) = 0$ is $a + b + c = 0$. This is a linear equation in abc space. Let's solve it. Treat a as a basic variable and b and c as free variables. Thus $a = -b - c$. Hence the most general $p(x)$ satisfying $p(1) = 0$ is

$p(x) = (-b - c)x^2 + bx + c = b(x - x^2) + c(1 - x^2)$

This shows that our space of $p(x)$ functions is spanned by $p_1(x) = x - x^2$ and $p_2(x) = 1 - x^2$. All we need to do is to show that p_1 and p_2 are linearly independent. If so, they form a basis for our set of functions and so the set has dimension 2. For linear independence we consider what linear combinations can equal the zero function:

$b(x - x^2) + c(1 - x^2) = 0 \Rightarrow$ when $x = 0$ we get $c = 0$ and when $x = -1$ we get $-2b = 0$. Thus the only linear combination giving the zero function is the zero combination. We are done

Problem 2.4.11

(a) We know that $P^{(3)}$ has dimension 4. Since we have been given 4 functions, they will form a basis if they are linearly independent. To show this we must show that

$c_1(1) + c_2(1 - t) + c_3(1 - t)^2 + c_4(1 - t)^3 = 0$

implies that all of the c_i's are zero. Expanding and collecting terms, we obtain

$c_1 + c_2(1 - t) + c_3(1 - 2t + t^2) + c_4(1 - 3t + 3t^2 - t^3) = 0$

$\Rightarrow (c_1 + c_2 + c_3 + c_4) + (-c_2 - 2c_3 - 3c_4)t + (c_3 + 3c_4)t^2 - c_4t^3 = 0$

$\Rightarrow c_1 + c_2 + c_3 + c_4 = 0, -c_2 - 2c_3 - 3c_4 = 0, c_3 + 3c_4 = 0, -c_4 = 0$

The last step is a consequence of the fact that the functions $1, t, t^2, t^3$ are linearly independent and so the coefficients in this linear combination of them must vanish. We have reduced to problem to that of solving 4 equations in 4 unknowns for the c_i's. A careful examination of these shows that all the c_i's are zero.

(b) Here we want c_i's such that

$c_1 + c_2(1 - t) + c_3(1 - 2t + t^2) + c_4(1 - 3t + 3t^2 - t^3) = 1 + t^3$

$\Leftrightarrow (c_1 + c_2 + c_3 + c_4) + (-c_2 - 2c_3 - 3c_4)t + (c_3 + 3c_4)t^2 - c_4t^3 = 1 + t^3$

$\Leftrightarrow c_1 + c_2 + c_3 + c_4 = 1, -c_2 - 2c_3 - 3c_4 = 0, c_3 + 3c_4 = 0, -c_4 = 1$

The solution of these equations (they are already in elimination form) is $c_4 = -1, c_3 = 3, c_2 = -3, c_1 = 2$. Here is a check:

$2 - 3(1 - t) + 3(1 - 2t + t^2) - (1 - 3t + 3t^2 - t^3) = 1 + t^3$

Problem 2.4.15

First set a general linear combo of the vectors (here 2x2 matrices) to zero:

$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = 0$

$\Leftrightarrow c_1 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} k & -3 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 0 \\ -k & 2 \end{pmatrix} + c_4 \begin{pmatrix} 0 & k \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} c_1 + kc_2 + c_3 & -c_1 - 3c_2 + kc_4 \\ c_2 - kc_3 - c_4 & 2c_3 - 2c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Leftrightarrow c_1 + kc_2 + c_3 = 0, -c_1 - 3c_2 + kc_4 = 0, c_2 - kc_3 - c_4 = 0, 2c_3 - 2c_4 = 0$

This has reduced our problem to one of solving a linear system. Here is the elimination:

$$\begin{pmatrix} 1 & k & 1 & 0 \\ -1 & -3 & 0 & k \\ 0 & 1 & -k & -1 \\ 0 & 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & k & 1 & 0 \\ 0 & k-3 & 1 & k \\ 0 & 1 & -k & -1 \\ 0 & 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & k & 1 & 0 \\ 0 & 1 & -k & -1 \\ 0 & 0 & 2 & -2 \\ 0 & k-3 & 1 & k \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & k & 1 & 0 \\ 0 & 1 & -k & -1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 + k(k-3) & 2k-3 \end{pmatrix} = \begin{pmatrix} 1 & k & 1 & 0 \\ 0 & 1 & -k & -1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & X \end{pmatrix}$$

where $X = 2k - 3 + 1 + k(k-3) = k^2 - k - 2 = (k-2)(k+1)$. For linear independence we need to conclude that all the c_i's are zero. This will be true if and only if the reduced matrix above has 4 pivots. It clearly has atleast 3. It will have 4 if $X \neq 0$. Thus k cannot be either 2 or -1.

Problem 2.5.3

(a) First let us find the kernel:

$$\begin{pmatrix} 1 & -3 & 2 & 0 \\ 2 & -6 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & -\frac{5}{2} \end{pmatrix}$$

So $w = 0, z = 0, x = 3y$, and therefore

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3y \\ y \\ 0 \\ 0 \end{pmatrix} = y \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

This shows that the kernel of A is the span of $\begin{pmatrix} 3 & 1 & 0 & 0 \end{pmatrix}^T$. The range is the span of the columns of A, or the most general vector of the form $A \begin{pmatrix} x & y & z & w \end{pmatrix}^T$. Thus

$$\text{rng } A = \left\{ \begin{pmatrix} x - 3y + 2z \\ 2x - 6y + 2w \\ z - 3w \end{pmatrix} \middle| \; x, y, z, \text{ and } w \text{ are arbitrary real numbers} \right\}$$

(b) In this case we do the elimination for the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & -3 & 2 & 0 & a \\ 2 & -6 & 0 & 2 & b \\ 0 & 0 & 1 & -3 & c \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -3 & 2 & 0 & a \\ 0 & 0 & -4 & 2 & b - 2a \\ 0 & 0 & 1 & -3 & c \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -3 & 2 & 0 & a \\ 0 & 0 & -4 & 2 & b - 2a \\ 0 & 0 & 0 & -\frac{5}{2} & c + \frac{1}{4}(b - 2a) \end{array}\right)$$

We note that the reduced system has 3 pivots and one free variable. Therefore there is always a solution of $Ax = \begin{pmatrix} a & b & c \end{pmatrix}^T$ regardless of the values of a, b, and c. Thus there are NO restrictions. In essence, the range of A is actually R^3.

Problem 2.5.4

(a) $\mathbf{b} = A\mathbf{x}^* = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

(b) First find the kernel of A:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $y = z$ and $x = y = z$ and so the kernel consists of the vectors

$$\mathbf{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The general solution of the initial system is then

$$\mathbf{x} = \mathbf{x}^* + \mathbf{z} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+z \\ 2+z \\ 3+z \end{pmatrix}.$$ Here is a check:

$$A\mathbf{x} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1+z \\ 2+z \\ 3+z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

Problems 2.5.21 (c)

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & 2 \\ 0 & 1 & 3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We conclude that the first two columns of A are a basis for the range of A since these are the columns corresponding to the pivots. For the range we note that
$y = -3z + 2w, x = -(-3z + 2w) - 2z - w = z - 3w$ and so

$$\mathbf{z} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} z - 3w \\ -3z + 2w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

and the two vectors on the right are a basis for the kernel of A. Now we do the same thing with A^T:

$$A^T = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & -1 & 7 \\ 1 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We conclude that the first two columns of A^T are a basis for the range of A^T, i.e. the corange of A, since these are the columns corresponding to the pivots. For the range we note that
$y = z, x = -y - 2z = -3z$ and so

$$\mathbf{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

and so the vector $\begin{pmatrix} -3 & 1 & 1 \end{pmatrix}^T$ is a basis for the kernel of A^T, i.e. the cokernel of A

Problem 2.5.23

First we do Gaussian elimination on

$$A = \begin{pmatrix} 1 & -3 & 0 \\ 2 & -6 & 4 \\ -3 & 9 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the pivots correspond to columns 1 and 3, a basis for the range of A consists of $\begin{pmatrix} 1 & 2 & -3 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 4 & 1 \end{pmatrix}^T$. To write the columns of A as linear combos of these, we just have to do this for column 2, so we do the following reduction:

$$\left(\begin{array}{cc|c} 1 & 0 & -3 \\ 2 & 4 & -6 \\ -3 & 1 & 9 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 4 & 0 \\ 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

and this implies that $\begin{pmatrix} -3 & -6 & 9 \end{pmatrix}^T = -3\begin{pmatrix} 1 & 2 & -3 \end{pmatrix}^T$. For the corange we do

$$A^T = \begin{pmatrix} 1 & 2 & -3 \\ -3 & -6 & 9 \\ 0 & 4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so a basis of the corange consists of $\begin{pmatrix} 1 & -3 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 2 & -6 & 4 \end{pmatrix}^T$. To write the columns of A^T as linear combos of these, we just have to do this for column 3, so we do the following reduction:

$$\left(\begin{array}{cc|c} 1 & 2 & -3 \\ -3 & -6 & 9 \\ 0 & 4 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 4 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

and this implies that $\begin{pmatrix} -3 & 9 & 1 \end{pmatrix}^T = c_1 \begin{pmatrix} 1 & -3 & 0 \end{pmatrix}^T + c_2 \begin{pmatrix} 2 & -6 & 4 \end{pmatrix}^T$ where $4c_2 = 1$, i.e. $c_2 = \frac{1}{4}$ and $c_1 = -3 - 2c_2 = -\frac{7}{2}$. Check:
$-\frac{7}{2}\begin{pmatrix} 1 & -3 & 0 \end{pmatrix}^T + \frac{1}{4}\begin{pmatrix} 2 & -6 & 4 \end{pmatrix}^T = \begin{pmatrix} -3 & 9 & 1 \end{pmatrix}^T$

Problem 2.6.2

(a) $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$ corresponds to the figure at the top of the page.

(b) Since the graph is connected, the kernel is spanned by $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$
For the cokernel we need to reduce

$$A^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \to \ldots \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and by back substitution we find that cokernel $A = \{0\}$, i.e. no circuits (obvious from the figure).
(c) None!

Problem 2.6.12

The incident matrix A for such a graph would be $n \times n$. We know (Proposition 2.51) that if the graph is connected, then the kernel of A is one dimensional and so A has rank $n-1$. This means that the rank of A^T is $n-1$ and so the cokernel of A is one dimensional, i.e. there is one circuit. If the graph is not connected, focus on a connected part of it and use a part of this subgraph with the same number of vertices and edges and apply the argument above.