Math 415 - Assignment 5 Solutions

Problems: 3.1.1, 3.1.3, 3.1.5, 3.1.14, 3.1.20, 3.2.37, 3.4.1, 3.4.2, 3.4.6, 3.4.22(vii), 3.4.24 (vii), 3.5.3, 3.5.6 (c)

Problem 3.1.1
Symmetry: $\langle \mathbf{v}, \mathbf{w} \rangle = v_1w_1 - v_1w_2 - v_2w_1 + bv_2w_2 = w_1v_1 - w_1v_2 - w_2v_1 + bw_2v_2 = \langle \mathbf{w}, \mathbf{v} \rangle$
Linearity:

$$\begin{aligned} \langle c\mathbf{v}+d\mathbf{u}, \mathbf{w} \rangle &= (cv_1 + du_1)w_1 - (cv_1 + du_1)w_2 - (cv_2 + du_2)w_1 + b(cv_2 + du_2)w_2 \\ &= cv_1w_1 + du_1w_1 - cv_1w_2 - du_1w_2 - cv_2w_1 - du_2w_1 + bcv_2w_2 + bdu_2w_2 \\ &= c(v_1w_1 - v_1w_2 - v_2w_1 + bv_2w_2) + d(u_1w_1 - u_1w_2 - u_2w_1 + bu_2w_2) \\ &= c\langle \mathbf{v}, \mathbf{w} \rangle + d\langle \mathbf{u}, \mathbf{w} \rangle \end{aligned}$$

Linearity in the second argument now follows from this calculation and symmetry.
Positivity: $\langle \mathbf{v}, \mathbf{v} \rangle = v_1^2 - v_1v_2 - v_2v_1 + bv_2^2 = v_1^2 - 2v_1v_2 + v_2^2 - v_2^2 + bv_2^2 = (v_1 - v_2)^2 + (b-1)v_2^2$
This expression is a sum of squares with positive coefficients when $b > 1$. Thus $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and vanishes only if $v_2 = 0$ and $v_1 = v_2$i.e. $\mathbf{v} = \mathbf{0}$. On the other hand, if $b \leq 1$, then any vector with $v_1 = v_2 \neq 0$ will make $\langle \mathbf{v}, \mathbf{v} \rangle$ either zero or negative.

Problem 3.1.3
Check positivity: $\langle \mathbf{v}, \mathbf{v} \rangle = v_1^2 + v_1v_2 + v_2v_1 + v_2^2 = (v_1 + v_2)^2$. This shows that the vector $\mathbf{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ is non-zero but $\langle \mathbf{v}, \mathbf{v} \rangle = 0$. Thus positivity is violated.

Problem 3.1.5
(a) Euclidean: $v_1^2 + v_2^2 = 1$. This is a circle of radius 1 in the $v_1$vs v_2 plane.
(b) Here $2v_1^2 + 5v_2^2 = 1$. This is an ellipse in the $v_1$vs v_2 plane of imnor axis $1/\sqrt{5}$ along the v_2 axis and major axis $1/\sqrt{2}$ along the v_1 axis.
(c) Here $v_1^2 - 2v_1v_2 + 4v_2^2 = (v_1 - v_2)^2 + 3v_2^2 = y_1^2 + 3y_2^2 = 1$ where $y_1 = v_1 - v_2$ and $y_2 = v_2$. This an ellipse in the $y_1$vs y_2 plane, a rotation of the $v_1$vs v_2 plane.
(d) See b) and c).

Problem 3.1.14
Using the dot product definition as $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$, we find that

$$\begin{aligned} \mathbf{v} \cdot (A\mathbf{w}) &= \mathbf{v}^T A\mathbf{w} \\ &= (A^T\mathbf{v})^T\mathbf{w} \\ &= (A^T\mathbf{v}) \cdot \mathbf{w} \end{aligned}$$

Problem 3.1.20
(a) $\langle x, 1+x^2 \rangle = \int_0^1 x(1+x^2)dx = \frac{3}{4}, \|x\| = \sqrt{\int_0^1 x^2dx} = \frac{1}{3}\sqrt{3}, \|1+x^2\| = \sqrt{\int_0^1 (1+x^2)^2dx} = \frac{2}{15}\sqrt{105}$
(b) $\langle x, 1+x^2 \rangle = \int_{-1}^1 x(1+x^2)dx = 0, \|x\| = \sqrt{\int_{-1}^1 x^2dx} = \frac{1}{3}\sqrt{6}, \|1+x^2\| = \sqrt{\int_{-1}^1 (1+x^2)^2dx} = \frac{2}{15}\sqrt{210}$
(c) $\langle x, 1+x^2 \rangle = \int_0^1 x(1+x^2)xdx = \frac{8}{15}, \|x\| = \sqrt{\int_0^1 x^2xdx} = \frac{1}{2}, \|1+x^2\| = \sqrt{\int_0^1 (1+x^2)^2xdx} = \frac{1}{6}\sqrt{42}$

Problem 3.2.37
(a) Cauchy Schwarz Inequality: $\left| \int_0^1 f(x)g(x)e^x dx \right| \leq \sqrt{\int_0^1 f(x)^2e^x dx}\sqrt{\int_0^1 g(x)^2e^x dx}$

Triangle Inequality: $\sqrt{\int_0^1 (f(x)+g(x))^2 e^x dx} \leq \sqrt{\int_0^1 f(x)^2 e^x dx} + \sqrt{\int_0^1 g(x)^2 e^x dx}$

(b) $\int_0^1 f(x)g(x)e^x dx = \int_0^1 e^{2x} dx = \frac{1}{2}e^2 - \frac{1}{2}, \int_0^1 f(x)^2 e^x dx = \int_0^1 e^x dx = e - 1, \int_0^1 g(x)^2 e^x dx = \int_0^1 e^{3x} dx = \frac{1}{3}e^3 - \frac{1}{3}, \int_0^1 (f(x)+g(x))^2 e^x dx = \int_0^1 (1+e^x)^2 e^x dx = \int_0^1 (e^x + 2e^{2x} + e^{3x}) dx = e + e^2 + \frac{1}{3}e^3 - \frac{7}{3}$

So Cauchy Schwarz: $\left|\frac{1}{2}e^2 - \frac{1}{2}\right| = 3.1945 \leq \sqrt{e-1}\sqrt{\frac{1}{3}e^3 - \frac{1}{3}} = 3.3063$ - verified!

and Triangle: $\sqrt{e + e^2 + \frac{1}{3}e^3 - \frac{7}{3}} = 3.8038 \leq \sqrt{e-1} + \sqrt{\frac{1}{3}e^3 - \frac{1}{3}} = 3.8331$ - verified!

(c) The angle between $f(x) = 1$ and $g(x) = e^x$ is θ where

$\cos\theta = \frac{\langle f,g\rangle}{\|f\|\|g\|} = \frac{\frac{1}{2}e^2 - \frac{1}{2}}{\sqrt{e-1}\sqrt{\frac{1}{3}e^3 - \frac{1}{3}}} = .9662 \Rightarrow \theta = \cos^{-1}(.9662) = .26074$ radians $= .26074\frac{180}{\pi} = 14.939$ degrees

Problem 3.4.1

(a) Yes: Diagonal with all positive entries

(b) No: Associated quadratic form is $q(x) = q(x_1, x_2) = 3x_1x_2$ which is negative any time that $x_1 < 0$ and $x_2 > 0$.

(c) No: The (1,1) entry is positive but the determinant is $1 \times 1 - 2 \times 2 = -3$ is negative.

(d) No: The (1,1) entry is positive but the determinant is $5 \times (-2) - 3 \times 3 = -19$ is negative.

(e) Yes: The (1,1) entry is positive and the determinant is $1 \times 3 - (-1) \times (-1) = 2$ is positive.

(f) No: The matrix is not symmetric and so cannot be positive definite.

Problem 3.4.2

$q(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 + 2x_2)\, x_1 + (2x_1 + 3x_2)\, x_2 = x_1^2 + 4x_1x_2 + 3x_2^2 = x_1^2 + 4x_1x_2 + 4x_2^2 - x_2^2 = (x_1 + 2x_2)^2 - x_2^2$. We see now that $q(1,0) = 1 > 0$ and $q(-2,1) = -1 < 0$.

Problem 3.4.6

The quadratic form of K is $q(x) = x^T K x$ and we know this is always positive unless $x = 0$. The quadratic form of cK is $\hat{q}(x) = x^T(cK)x = c(x^T K x) = cq(x)$. Given that $c > 0$ we conclude that $\hat{q}(x)$ is positive unless $x = 0$.

Problem 3.4.22(vii)

$$(a)\ K = \begin{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ -4 \\ 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ -1 \\ -2 \end{pmatrix} \\ \begin{pmatrix} -2 & 1 & -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} & \begin{pmatrix} -2 & 1 & -4 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ -4 \\ 3 \end{pmatrix} & \begin{pmatrix} -2 & 1 & -4 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ -1 \\ -2 \end{pmatrix} \\ \begin{pmatrix} -1 & 3 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} & \begin{pmatrix} -1 & 3 & -1 & -2 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ -4 \\ 3 \end{pmatrix} & \begin{pmatrix} -1 & 3 & -1 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ -1 \\ -2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 30 & 0 & -6 \\ 0 & 30 & 3 \\ -6 & 3 & 15 \end{pmatrix}$$

(b) By Gaussian Elimination:

$$\begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ 3 & -4 & -1 \\ 4 & 3 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 2 & 2 \\ 0 & 11 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are three pivots, the vectors are linearly independent, and so K is positive definite.

(c) Nothing to do! $\ker K = \{\mathbf{0}\}$

Problem 3.4.24 (vii)

The new inner product has the form $\langle x,y\rangle = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

so $k_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 10$

$k_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -4 \\ 3 \end{pmatrix} = -2$

$k_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ -1 \\ -2 \end{pmatrix} = -1$

$k_{22} = \begin{pmatrix} -2 & 1 & -4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -4 \\ 3 \end{pmatrix} = \frac{145}{12}$

$k_{23} = \begin{pmatrix} -2 & 1 & -4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ -1 \\ -2 \end{pmatrix} = \frac{10}{3}$

$k_{33} = \begin{pmatrix} -1 & 3 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ -1 \\ -2 \end{pmatrix} = \frac{41}{6}$

So $K = \begin{pmatrix} 10 & -2 & -1 \\ -2 & \frac{145}{12} & \frac{10}{3} \\ -1 & \frac{10}{3} & \frac{41}{6} \end{pmatrix}$

Problem 3.5.3

(a) By Gaussian elimination

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & c-1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & c-1 & 1 \\ 0 & 0 & \frac{c-2}{c-1} \end{pmatrix}$$

We need all diagonal elements to be positive, so we need $c > 2$.

(b) When $c = 3$ the above calculation gives us

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

so $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

(c) So if we set $\begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+\frac{1}{2}z \\ z \end{pmatrix}$

then $q(x,y,z) = \hat{q}(r,s,t) = r^2 + 2s^2 + \frac{1}{2}t^2$ which is a sum of squares

(d) Since $q(x,y,z) = \hat{q}(r,s,t)$ and the transformation $\begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} x+y \\ y+\frac{1}{2}z \\ z \end{pmatrix}$ is invertible, K is positive definite iff $q(x,y,z)$ is positive definite, iff $\hat{q}(r,s,t)$ is positive define (which is true since it is a sum of squares with positive coefficients).

Problem 3.5.6 (c)

$$\begin{aligned}
q(x_1,x_2,x_3) &= 2x_1^2 + x_1x_2 - 2x_1x_3 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \\
&= 2(x_1^2 + 2x_1(\frac{x_2-2x_3}{4})) + 2x_2^2 - 2x_2x_3 + 2x_3^2 \\
&= 2(x_1^2 + 2x_1(\frac{x_2-2x_3}{4}) + (\frac{x_2-2x_3}{4})^2) - 2(\frac{x_2-2x_3}{4})^2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \\
&= 2(x_1 + \frac{x_2-2x_3}{4})^2 - \frac{1}{8}x_2^2 + \frac{1}{2}x_2x_3 - \frac{1}{2}x_3^2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \\
&= 2(x_1 + \frac{x_2-2x_3}{4})^2 + \frac{15}{8}x_2^2 - \frac{3}{2}x_2x_3 + \frac{3}{2}x_3^2 \\
&= 2(x_1 + \frac{x_2-2x_3}{4})^2 + \frac{15}{8}(x_2^2 - 2x_2(\frac{2}{5}x_3)) + \frac{3}{2}x_3^2 \\
&= 2(x_1 + \frac{x_2-2x_3}{4})^2 + \frac{15}{8}(x_2^2 - 2x_2(\frac{2}{5}x_3) + (\frac{2}{5}x_3)^2) - \frac{15}{8}(\frac{2}{5}x_3)^2 + \frac{3}{2}x_3^2 \\
&= 2(x_1 + \frac{x_2-2x_3}{4})^2 + \frac{15}{8}(x_2 - \frac{2}{5}x_3)^2 + \frac{6}{5}x_3^2
\end{aligned}$$

which is a sum of squares with positive coefficients. Moreover, it vanishes only if $x_3 = 0, x_2 = \frac{2}{5}x_3 = 0, x_1 = -\frac{1}{4}(x_2 - 2x_3) = 0$. Done!