

Math 415 - Assignment 6 Solutions

Problems: 4.2.1, 4.2.4, 4.2.5 (c), 4.2.7 (b), 4.3.1, 4.3.3, 4.3.14 (d), 4.3.17, 4.4.3

Problem 4.2.1

First we write this quadratic function in the standard form $f(x) = x^T K x - 2x^T f + c$ where

$$K = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}, f = \begin{pmatrix} 1 \\ 0 \\ -\frac{3}{2} \end{pmatrix}, c = 2. \text{ By Gaussian elimination}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = U$$

and since the diagonal entries of U are all positive, K is positive definite. This shows that the minimizer is unique. By Theorem 4.1 the minimizer is the solution of $Kx = f$ and we solve this by elimination:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -\frac{3}{2} \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & 1 & -\frac{3}{2} \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & -1 \end{array} \right)$$

Therefore $\frac{1}{2}z = -1$ or $z = -2$, $2y = -1 - z$ or $y = \frac{1}{2}(-1 + 2) = \frac{1}{2}$, $x = 1 - y = 1 - \frac{1}{2} = \frac{1}{2}$. Then the minimum value is

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -2 \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -2 \end{pmatrix} - 2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -\frac{3}{2} \end{pmatrix} + 2 = -\frac{3}{2}$$

Problem 4.2.4

(a) Since the (1,1) entry of A is positive, we just need the determinant to be positive: $4 - b^2 > 0$, i.e. $-2 < b < 2$

$$(b) A = \begin{pmatrix} 1 & b \\ b & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & b \\ 0 & 4 - b^2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 4 - b^2 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$\text{so } D = \begin{pmatrix} 1 & 0 \\ 0 & 4 - b^2 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

(c) This part of the question is an application of Theorem 4.4. First assume that b is such that A is positive definite. Then the unique minimizer satisfies $Ax = f = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$. Thus $x = A^{-1}f =$

$$\begin{pmatrix} 1 & b \\ b & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{-4+b^2} & \frac{b}{-4+b^2} \\ \frac{b}{-4+b^2} & -\frac{1}{-4+b^2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{b}{-4+b^2} \\ -\frac{1}{-4+b^2} \end{pmatrix}$$

and so the minimum is $\left(\frac{b}{-4+b^2}\right)^2 + 2b\left(\frac{b}{-4+b^2}\right)\left(-\frac{1}{-4+b^2}\right) + 4\left(-\frac{1}{-4+b^2}\right)^2 - 2\left(-\frac{1}{-4+b^2}\right) = \frac{1}{-4+b^2}$.

If $b > 2$ or $b < -2$, A is neither definite or semidefinite, and so the minimum in this case is $-\infty$.

If $b = 2$, the quadratic function reduces to $p(x, y) = (x + 2y)^2 - 2y$ and so $p(-2y, y) = -2y \rightarrow -\infty$

as $y \rightarrow \infty$. If $b = -2$, the quadratic function reduces to $p(x, y) = (x - 2y)^2 - 2y$ and so $p(2y, y) = -2y \rightarrow -\infty$ as $y \rightarrow \infty$. Done!

Problem 4.2.5 (c)

$$p(x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} - 3$$

$= 3x^2 - 2xy + 2xz + 2y^2 - 2yz + 3z^2 - 2x + 4z - 3$. Now let's solve $Kx = f$:

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ -1 & 2 & -1 & 0 \\ 1 & -1 & 3 & -2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & \frac{5}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{8}{3} & -\frac{7}{3} \end{array} \right) \rightarrow$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 5 & -2 & 1 \\ 0 & -\frac{2}{3} & \frac{8}{3} & -\frac{7}{3} \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 12 & -11 \end{array} \right)$$

so $12z = -11$, i.e. $z = -\frac{11}{12}$, $5y = 1 + 2z$, i.e. $y = \frac{1}{5}(1 - \frac{22}{12}) = -\frac{1}{6}$, $3x = 1 + y - z$, i.e. $x =$

$\frac{1}{3}(1 - \frac{1}{6} + \frac{11}{12}) = \frac{7}{12}$. The upper diagonal reduced matrix has only positive entries on the diagonal, and therefore K is positive definite. Hence there is a unique minimizer as given above, and the minimum is

$$\begin{pmatrix} \frac{7}{12} \\ -\frac{1}{6} \\ -\frac{11}{12} \end{pmatrix}^T \begin{pmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{7}{12} \\ -\frac{1}{6} \\ -\frac{11}{12} \end{pmatrix} - 2 \begin{pmatrix} \frac{7}{12} \\ -\frac{1}{6} \\ -\frac{11}{12} \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} - 3 = -\frac{65}{12}$$

Problem 4.2.7 (b)

To maximize a quadratic function, you minimize the negative of the function, i.e. minimize $p(x, y) = 2x^2 - 6xy + 3y^2 - 4x + 3y =$

$$\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 2 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 2 \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix}$$

But the K matrix here is indefinite, i.e. the (1,1) entry is positive but the determinant is negative: $2 \times 3 - (-3)^2 = -3$. Thus the minimum of p is $-\infty$, and so the maximum of the original quadratic function is ∞ .

Problem 4.3.1

This is a standard closest point problem. Let A be the matrix with the two spanning vectors as columns and define $K = A^T A$, $f = A^T b$ and $c = \|b\|^2$. This gives us

$$K = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 3 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & -5 \\ -5 & 10 \end{pmatrix}, f = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 3 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, c =$$

$$\|b\|^2 = 3$$

So now we need to solve $Kx = f$, i.e. $x = A^{-1}f$:

$$x = \begin{pmatrix} 6 & -5 \\ -5 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{1}{35} \\ \frac{1}{7} & \frac{6}{35} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ \frac{22}{35} \end{pmatrix}$$

The closest point is the vector

$$v = \frac{6}{7} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \frac{22}{35} \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ \frac{38}{35} \\ \frac{36}{35} \end{pmatrix}$$

and the minimum distance is

$$\begin{pmatrix} \frac{6}{7} \\ \frac{22}{35} \end{pmatrix}^T \begin{pmatrix} 6 & -5 \\ -5 & 10 \end{pmatrix} \begin{pmatrix} \frac{6}{7} \\ \frac{22}{35} \end{pmatrix} - 2 \begin{pmatrix} \frac{6}{7} \\ \frac{22}{35} \end{pmatrix}^T \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 3 = \frac{1}{35}$$

Problem 4.3.3

First we need to find a basis spanning the plane: $x = -2y + z$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y + z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Now we repeat the calculation in 4.3.1 using these two spanning vectors:

$$K = \begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}, f = \begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = \|b\|^2 =$$

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So now we need to solve $Kx = f$, i.e. $x = A^{-1}f$:

$$x = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{6} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{5}{6} \end{pmatrix}$$

The closest point is the vector

$$v = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \\ \frac{5}{6} \end{pmatrix}$$

and the minimum distance is

$$\begin{pmatrix} \frac{1}{3} \\ \frac{5}{6} \end{pmatrix}^T \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{5}{6} \end{pmatrix} - 2 \begin{pmatrix} \frac{1}{3} \\ \frac{5}{6} \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 = \frac{1}{6}$$

Problem 4.3.14 (d)

First write the linear system in matrix form $Ax = b$. This implies that

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 0 & 1 & -3 \\ -5 & 2 & 1 \end{pmatrix}, b = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 3 \end{pmatrix}$$

Now we repeat the calculations in problem 4.3.1:

$$K = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 0 & 1 & -3 \\ -5 & 2 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 0 & 1 & -3 \\ -5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 30 & -12 & 0 \\ -12 & 6 & -4 \\ 0 & -4 & 20 \end{pmatrix}$$

$$f = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 0 & 1 & -3 \\ -5 & 2 & 1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -14 \\ 5 \\ 7 \end{pmatrix}, c = \|b\|^2 = 11$$

So now we need to solve $Kx = f$, i.e. $x = A^{-1}f$:(use Gaussian elimination, not inverses as here)

$$x = \begin{pmatrix} 30 & -12 & 0 \\ -12 & 6 & -4 \\ 0 & -4 & 20 \end{pmatrix}^{-1} \begin{pmatrix} -14 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{13}{30} & 1 & \frac{1}{5} \\ 1 & \frac{5}{2} & \frac{3}{20} \\ \frac{1}{5} & \frac{1}{2} & \frac{3}{20} \end{pmatrix} \begin{pmatrix} -14 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 2 \\ \frac{3}{4} \end{pmatrix}$$

The least squares solution is the x above and the squared error involved is

$$\begin{pmatrix} \frac{1}{3} \\ 2 \\ \frac{3}{4} \end{pmatrix}^T \begin{pmatrix} 30 & -12 & 0 \\ -12 & 6 & -4 \\ 0 & -4 & 20 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 2 \\ \frac{3}{4} \end{pmatrix} - 2 \begin{pmatrix} \frac{1}{3} \\ 2 \\ \frac{3}{4} \end{pmatrix}^T \begin{pmatrix} -14 \\ 5 \\ 7 \end{pmatrix} + 11 = \frac{5}{12}$$

Problem 4.3.17

As in the previous problem we have

$$K = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{pmatrix}$$

$$f = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 5 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} -18 \\ 28 \\ 17 \end{pmatrix}, c = \|b\|^2 = 125$$

So now we need to solve $Kx = f$, i.e. $x = A^{-1}f$:(use Gaussian elimination, not inverses as here)

$$x = \begin{pmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{pmatrix}^{-1} \begin{pmatrix} -18 \\ 28 \\ 17 \end{pmatrix} = \begin{pmatrix} \frac{132}{1171} & \frac{6}{1171} & \frac{61}{1171} \\ \frac{6}{1171} & \frac{107}{1171} & \frac{y=56}{1171} \\ \frac{61}{1171} & \frac{2342}{1171} & \frac{1171}{1171} \end{pmatrix} \begin{pmatrix} -18 \\ 28 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

The least squares solution is the x above and the squared error involved is

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}^T \begin{pmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}^T \begin{pmatrix} -18 \\ 28 \\ 17 \end{pmatrix} + 125 = 0$$

This indicates that the vector b lies in the span of the columns of A , i.e. that the x we have found is actually a solution of $Ax = b$.

Problem 4.4.3

(a) This is a straight forward use of results that have already been developed. Let t be measured from 1989, that is $t = 0$ corresponds to 1989 and $t = 10$ corresponds to 1999. First we calculate the following averages:

$$\bar{t} = \frac{1}{11}(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) = 5$$

$$\bar{t}^2 = \frac{1}{11}(0 + 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100) = 35$$

$$\bar{y} = \frac{1}{11}(86.4 + 89.8 + 92.8 + 96.0 + 99.6 + 103.1 + 106.3 + 109.5 + 113.3 + 120.0 + 129.5) = 104.21$$

$$\bar{ty} = \frac{1}{11}(86.4(0) + 89.8(1) + 92.8(2) + 96.0(3) + 99.6(4) + 103.1(5) + 106.3(6) + 109.5(7) + 113.3(8) +$$

$$120.0(9) + 129.5(10) = 560.27$$

By equation (4.38) we have $\beta = (\overline{ty} - \bar{t}\bar{y})/(\overline{t^2} - (\bar{t})^2) = (560.27 - (5)(104.21))/(35 - 25) = 3.922$, $\alpha = \bar{y} - \bar{t}\beta = 104.21 - 5(3.922) = 84.6$. Thus the least squares line for these data is $y = 84.6 + 3.922t$.

(b) The year 2005 corresponds to $t = 16$, so the median price in that year should be $y = 84.6 + 3.922(16) = 147.35$

The year 2010 corresponds to $t = 21$, so the median price in that year should be $y = 84.6 + 3.922(21) = 166.96$