Math 415 - Assignment 7 Solutions

Problems: 5.1.2, 5.1.5, 5.1.6, 5.1.23, 5.1.27, 5.2.2 (a), 5.2.4 (b), 5.2.6 (d), 5.3.1, 5.3.2, 5.3.7, 5.3.8

Problem 5.1.2

(a) Basis: the first vector and third vector are not orthogonal.

(b) Othonormal basis: Since $v_1 \cdot v_1 = \left(-\frac{4}{13}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(-\frac{48}{65}\right)^2 = 1, v_2 \cdot v_2 = \left(\frac{12}{13}\right)^2 + \left(-\frac{5}{13}\right)^2 = 1, v_3 \cdot v_3 = \left(\frac{3}{13}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{36}{65}\right)^2 = 1$, we see that each vector is a unit vector. As for orthogonality, we have $v_1 \cdot v_2 = \left(-\frac{4}{13}\right)\left(\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{0}{1}\right) + \left(-\frac{48}{65}\right)\left(-\frac{5}{13}\right) = 0, v_1 \cdot v_3 = \left(-\frac{4}{13}\right)\left(\frac{3}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{4}{5}\right) + \left(-\frac{48}{65}\right)\left(\frac{36}{65}\right) = 0, v_2 \cdot v_3 = \left(\frac{12}{13}\right)\left(\frac{3}{13}\right) + (0)\left(\frac{4}{5}\right) + \left(-\frac{5}{13}\right)\left(\frac{36}{65}\right) = 0$.

(c) Basis: the first vector and second vector are not orthogonal.

Problem 5.1.5

(a) In this case we need $\begin{pmatrix} a \\ 1 \end{pmatrix}^T \begin{pmatrix} -a \\ 1 \end{pmatrix} = -a^2 + 1 = 0$, i.e. $a = \pm 1$

(b) In this case we need $\begin{pmatrix} a \\ 1 \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -a \\ 1 \end{pmatrix} = -3a^2 + 2 = 0$, i.e. $a = \pm\sqrt{\frac{2}{3}}$

(c) In this case we need $\begin{pmatrix} a \\ 1 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -a \\ 1 \end{pmatrix} = -(2a-1)a - a + 3 = -2a^2 + 3 = 0$, i.e. $a = \pm\sqrt{\frac{3}{2}}$

Problem 5.1.6

Here we need $\begin{pmatrix} 1 \\ 2 \end{pmatrix}^T \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -a + 2b = 0$, so $a = 2b$. In addition the matrix $K = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ must be positive definite in order to define an inner product, so $a > 0$ and $\det A = ab > 0$. We conclude that $a = 2b$ with b being any positive number.

Problem 5.1.23

(a) $\langle v_1, v_2 \rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0$. Done!

(b) $\|v_1\|^2 = \langle v_1, v_1 \rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3$

$\|v_2\|^2 = \langle v_2, v_2 \rangle = \begin{pmatrix} -2 \\ 1 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 15$

$\langle v, v_1 \rangle = \begin{pmatrix} 3 \\ 2 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 7$

$\langle v, v_2 \rangle = \begin{pmatrix} 3 \\ 2 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -5$

and so $v = av_1 + bv_2$ where $a = \frac{7}{3}$ and $b = \frac{-5}{15}$

(c) $\|v\|^2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 18$, on the other hand, $a^2\|v_1\|^2 + b^2\|v_2\|^2 = 3\left(\frac{7}{3}\right)^2 + 15\left(\frac{-5}{15}\right)^2 = 18$. Done!

(d) $u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, u_2 = \frac{1}{\sqrt{15}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(e) $v = cu_1 + du_2$ where $c = \langle v, u_1 \rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 \\ 2 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{7}{3}\sqrt{3}$

and $d = \langle v, u_2 \rangle = \frac{1}{\sqrt{15}}\begin{pmatrix} 3 \\ 2 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -\frac{1}{3}\sqrt{15}$. Then $c^2 + d^2 = \left(\frac{7}{3}\sqrt{3}\right)^2 + \left(-\frac{1}{3}\sqrt{15}\right)^2 = 18 = \|v\|^2$. Done!

Problem 5.1.27

(a) $\langle P_0, P_1\rangle = \int_0^1 1(t-\frac{2}{3})tdt = 0, \langle P_0, P_2\rangle = \int_0^1 1(t^2-\frac{6}{5}t+\frac{3}{10})tdt = 0, \langle P_1, P_2\rangle = \int_0^1 (t-\frac{2}{3})(t^2-\frac{6}{5}t+\frac{3}{10})tdt = 0$, so the vectors are mutually orthogonal.

(b) $\langle P_0, P_0\rangle = \int_0^1 1tdt = \frac{1}{2}, \langle P_1, P_1\rangle = \int_0^1 (t-\frac{2}{3})^2 tdt = \frac{1}{36}, \langle P_2, P_2\rangle = \int_0^1 (t^2-\frac{6}{5}t+\frac{3}{10})^2 tdt = \frac{1}{600}$, so the corresponding orthonormal basis is

$Q_0(t) = \sqrt{2}, Q_1(t) = 6(t-\frac{2}{3}), Q_2(t) = 10\sqrt{6}(t^2-\frac{6}{5}t+\frac{3}{10})$.

(c) $a = \langle t^2, Q_0(t)\rangle = \int_0^1 t^2\sqrt{2}tdt = \frac{1}{4}\sqrt{2}, b = \langle t^2, Q_1(t)\rangle = \int_0^1 t^2 6(t-\frac{2}{3})tdt = \frac{1}{5}, c = \langle t^2, Q_2(t)\rangle = \int_0^1 t^2 10\sqrt{6}(t^2-\frac{6}{5}t+\frac{3}{10})tdt = \frac{1}{60}\sqrt{6}$ so

$t^2 = \frac{1}{4}\sqrt{2}Q_0(t) + \frac{1}{5}Q_1(t) + \frac{1}{60}\sqrt{6}Q_2(t)$

Problem 5.2.2 (a)

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, w_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, w_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Set $v_1 = w_1, \|v_1\|^2 = 2$

$$\text{Then } v_2 = w_2 - \frac{w_2\cdot v_1}{\|v_1\|^2}v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} - \frac{0}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \|v_2\|^2 = 2$$

$$v_3 = w_3 - \frac{w_3\cdot v_1}{\|v_1\|^2}v_1 - \frac{w_3\cdot v_2}{\|v_2\|^2}v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \|v_3\|^2 = 1$$

$$v_4 = w_4 - \frac{w_4\cdot v_1}{\|v_1\|^2}v_1 - \frac{w_4\cdot v_2}{\|v_2\|^2}v_2 - \frac{w_4\cdot v_3}{\|v_3\|^2}v_3$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{0}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{1}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \|v_4\|^2 = 1$$

Thus our orthonormal basis is

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, u_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, u_4 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Problem 5.2.4 (b)

First find a basis for the subspace defined by the plane: $2x - y + 3z = 0 \Rightarrow x = \frac{1}{2}y - \frac{3}{2}z$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y - \frac{3}{2}z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$$

Therefore, set

$$w_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$$

Then we define

$v_1 = w_1, \|v_1\|^2 = \frac{5}{4}$

$$v_2 = w_2 - \frac{w_2\cdot v_1}{\|v_1\|^2}v_1 = \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix} - \frac{-\frac{3}{4}}{\frac{5}{4}}\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ \frac{3}{5} \\ 1 \end{pmatrix}, \|v_2\|^2 = \frac{70}{25} = \frac{14}{5}$$

Therefore an orthonormal basis of the plane is

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} -\frac{6}{\sqrt{70}} \\ \frac{3}{\sqrt{70}} \\ \frac{5}{\sqrt{70}} \end{pmatrix}$$

Problem 5.2.6 (d)

First we need to find a basis for the range of the matrix:

$$\begin{pmatrix} 1 & -2 & 2 \\ 2 & -4 & 1 \\ 0 & 0 & -1 \\ -2 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ Thus our basis is columns 1 and 3:

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix}, w_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 5 \end{pmatrix}$$

Then we define

$v_1 = w_1, \|v_1\|^2 = 5$

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 5 \end{pmatrix} - \frac{-6}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{16}{5} \\ \frac{17}{5} \\ -1 \\ \frac{13}{5} \end{pmatrix}, \|v_2\|^2 = \frac{739}{25}$$

Therefore an orthonormal basis of the range of the matrix is

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{pmatrix}, u_2 = \begin{pmatrix} \frac{16}{\sqrt{739}} \\ \frac{17}{\sqrt{739}} \\ -\frac{5}{\sqrt{739}} \\ \frac{13}{\sqrt{739}} \end{pmatrix}$$

Problem 5.3.1

(a) Not orthogonal:

$$\det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \neq 1$$

(b) Proper orthogonal:

$$\begin{pmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{pmatrix}^T \begin{pmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \det \begin{pmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{pmatrix} = 1$$

(c) Orthogonal:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^T \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \det \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1$$

(d) Proper orthogonal:

$$\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}^T \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \det \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} = 1$$

(e) Not orthogonal:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix}^T \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{61}{144} & \frac{3}{10} & \frac{7}{30} \\ \frac{3}{10} & \frac{769}{3600} & \frac{1}{6} \\ \frac{7}{30} & \frac{1}{6} & \frac{469}{3600} \end{pmatrix}$$

(f) Not orthogonal:

$$\begin{pmatrix} \frac{3}{3} & 0 & \frac{4}{5} \\ -\frac{4}{13} & \frac{12}{13} & \frac{3}{13} \\ -\frac{48}{65} & -\frac{5}{13} & \frac{36}{65} \end{pmatrix}^T \begin{pmatrix} \frac{3}{3} & 0 & \frac{4}{5} \\ -\frac{4}{13} & \frac{12}{13} & \frac{3}{13} \\ -\frac{48}{65} & -\frac{5}{13} & \frac{36}{65} \end{pmatrix} = \begin{pmatrix} \frac{41}{25} & 0 & \frac{8}{25} \\ 0 & 1 & 0 \\ \frac{8}{25} & 0 & 1 \end{pmatrix}$$

Problem 5.3.2

(a) $$R^T R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q^T Q = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) We compute RQ and QR here and go through the exercise above:

$$RQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}^T \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$QR = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Since $\det R = -1, \det Q = 1, \det RQ = \det R \det Q = -1 = \det Q \det R = \det QR$, we see that only Q is proper orthogonal.

Problem 5.3.7

If Q_1 and Q_2 are proper orthogonal and R_1 and R_2 are not proper orthogonal, then $\det Q_1 = \det Q_2 = 1$ and $\det R_1 = \det R_2 = -1$. Thus $\det Q_1Q_2 = \det Q_1 \det Q_2 = 1$, $\det Q_1R_1 = \det Q_1 \det R_1 = -1$, and $\det R_1R_2 = \det R_1 \det R_2 = -1 \times -1 = 1$. Thus Q_1Q_2 and R_1R_2 are proper and Q_1R_1 is not.

Problem 5.3.8

(a) Set $P = Q^T$. Then $P^TP = (Q^T)^T(Q^T) = QQ^T = QQ^{-1} = I$, so P is orthogonal.

(b) The columns of P are orthonormal and so the rows of $Q = P^T$ are orthonormal by part (a).