Math 415 - Assignment 8 Solutions

Problems: 5.5.1 (f).5.5.3, 5.5.7, 5.5.11 (b), 5.6.1 (a), 5.6.3 (b), 5.6.5 for 5.6.1 (a), 5.6.7

Problem 5.5.1 (f)

We first need to find a basis of the cokernel:

$$A^T = \begin{pmatrix} -1 & 0 & 3 \\ 2 & 1 & -2 \\ 3 & 1 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

so $x = 3z, y = -4z$ and therefore the cokernel is spanned by $\begin{pmatrix} 3 & -4 & 1 \end{pmatrix}^T$. By inspection none of the vectors is orthogonal to the cokernel

Problem 5.5.3

First we need to find a basis for the range:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & -2 \\ 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 \\ 0 & -\frac{10}{3} \\ 0 & \frac{4}{3} \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 \\ 0 & -\frac{10}{3} \\ 0 & 0 \end{pmatrix}$$ so the columns of A are a basis. Now set

$w_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$ and apply Gram Schmidt:

$v_1 = w_1, \|v_1\|^2 = 14$

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} - \frac{4}{14} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{8}{7} \\ -\frac{18}{7} \\ \frac{12}{7} \end{pmatrix}, \|v_2\|^2 = \frac{76}{7}$$

Therefore an orthonormal basis of the range is

$$u_1 = \begin{pmatrix} \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix}, u_2 = \begin{pmatrix} \frac{4}{\sqrt{133}} \\ -\frac{9}{\sqrt{133}} \\ \frac{6}{\sqrt{133}} \end{pmatrix}.$$ Now the orthogonal projection is

$$w = c_1 u_1 + c_2 u_2 \text{ where } c_1 = v \cdot u_1 = \begin{pmatrix} \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{5}{7}\sqrt{14}$$

and $c_2 = v \cdot u_2 = \begin{pmatrix} \frac{4}{\sqrt{133}} \\ -\frac{9}{\sqrt{133}} \\ \frac{6}{\sqrt{133}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{4}{\sqrt{133}}$. Thus

$$w = c_1 u_1 + c_2 u_2 = \frac{5}{7}\sqrt{14} \begin{pmatrix} \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} + \frac{4}{\sqrt{133}} \begin{pmatrix} \frac{4}{\sqrt{133}} \\ -\frac{9}{\sqrt{133}} \\ \frac{6}{\sqrt{133}} \end{pmatrix} = \begin{pmatrix} \frac{43}{19} \\ \frac{22}{19} \\ \frac{17}{19} \end{pmatrix}$$

Problem 5.5.7

Set $v = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}, w_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 5 \end{pmatrix}, w_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$. Now apply Gram Schmidt

First we have $v_1 = w_1, \|v_1\|^2 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 5 \end{pmatrix}^T \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 5 \end{pmatrix} = 40$

$$\langle w_2, v_1 \rangle = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 5 \end{pmatrix} = 0$$

Then $v_2 = w_2 - \frac{w_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \|v_2\|^2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} = 20$

Let's write the projection as $w = a_1 v_1 + a_2 v_2$ where $a_1 = \frac{\langle v, v_1 \rangle}{\|v_1\|^2}$ and $a_2 = \frac{\langle v, v_2 \rangle}{\|v_2\|^2}$. First compute the inner products:

$$\langle v, v_1 \rangle = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}^T \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 5 \end{pmatrix} = 4,$$

$$\langle v, v_2 \rangle = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}^T \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} = 12$$

Thus

$$w = a_1 v_1 + a_2 v_2 = \frac{4}{40} \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix} + \frac{12}{20} \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{13}{10} \\ \frac{4}{5} \\ -\frac{1}{10} \\ -\frac{2}{5} \end{pmatrix}$$

Problem 5.5.11 (b)

Begin with the system

$$\begin{pmatrix} 3 & -1 \\ 0 & 2 \\ -2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

and set

$$v = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 5 \end{pmatrix}$$

Next note that the columns of the coefficient matrix are orthogonal in the Euclidean inner product. Therefore the least squares solution is the orthogonal projection onto the span of the two columns. Since

$$\|v_1\|^2 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix}^T \begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix} = 14, \|v_2\|^2 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 5 \end{pmatrix}^T \begin{pmatrix} -1 \\ 2 \\ 1 \\ 5 \end{pmatrix} = 31,$$

$$v \cdot v_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix} = 9, v \cdot v_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 2 \\ 1 \\ 5 \end{pmatrix} = 4$$

the least squares solution is

$$\begin{pmatrix} 3 & -1 \\ 0 & 2 \\ -2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \\ 1 \\ 5 \end{pmatrix}$$

where $x = \frac{v \cdot v_1}{\|v_1\|^2} = \frac{9}{14}, y = \frac{v \cdot v_2}{\|v_2\|^2} = \frac{4}{31}$.

Problem 5.6.1 (a)

A vector $v = (x, y, z)^T$ is in $W^\perp$ if $(x, y, z) \cdot (3, -1, 1) = 3x - y + z = 0$. Thus $x = \frac{1}{3}y - \frac{1}{3}z$ and so

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3}y - \frac{1}{3}z \\ y \\ z \end{pmatrix} = \frac{1}{3}y \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} - \frac{1}{3}z \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ and so

$W^{\perp} = \text{span}\{(1,3,0)^T, (1,0,-3)^T\}$. Its dimension is 2.

Problem 5.6.3 (b)

A vector $v = (x,y,z,w)^T$ is in $W^{\perp}$ if
$(x,y,z,w) \cdot (1,2,-1,3) = x + 2y - z + 3w = 0$
$(x,y,z,w) \cdot (-2,0,1,-2) = -2x + z - 2w = 0$
$(x,y,z,w) \cdot (-1,2,0,1) = -x + 2y + w = 0$. Here is elimination:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ -2 & 0 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 4 & -1 & 4 \\ 0 & 4 & -1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 4 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so z and w are free and $y = \frac{1}{4}z - w, x = -2y + z - 3w = -2(\frac{1}{4}z - w) + z - 3w = \frac{1}{2}z - w$. Hence

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z - w \\ \frac{1}{4}z - w \\ z \\ w \end{pmatrix} = \frac{1}{4}z \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} - w \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \text{ and so}$$

$W^{\perp} = \text{span}\{(2,1,1,0)^T, (1,1,0,-1)^T\}$

Problem 5.6.5 (for 5.6.1 (a))

A vector $v = (x,y,z)^T$ is in $W^{\perp}$ if $\langle (x,y,z)^T, (3,-1,1)^T \rangle = 3x - 2y + 3z = 0$. Thus $x = \frac{2}{3}y - z$ and so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3}y - z \\ y \\ z \end{pmatrix} = \frac{1}{3}y \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ and so}$$

$W^{\perp} = \text{span}\{(2,3,0)^T, (-1,0,1)^T\}$

Problem 5.6.7

(a) One basis of $P^{(2)}$ consists of $p_0(x) = 1, p_1(x) = x, p_2(x) = x^2$, so we want a general member $p(x)$ in $P^{(4)}$ to be orthogonal to these:
$\int_{-1}^{1} p(x)dx = 0, \int_{-1}^{1} xp(x)dx = 0, \int_{-1}^{1} x^2p(x)dx = 0$
(b) Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$. Then
$\int_{-1}^{1}(ax^4 + bx^3 + cx^2 + dx + e)dx = \frac{2}{5}a + \frac{2}{3}c + 2e = 0$
$\int_{-1}^{1} x(ax^4 + bx^3 + cx^2 + dx + e)dx = \frac{2}{5}b + \frac{2}{3}d$
$\int_{-1}^{1} x^2(ax^4 + bx^3 + cx^2 + dx + e)dx = \frac{2}{7}a + \frac{2}{5}c + \frac{2}{3}e$
Let's get rid of the fractions by multiplying the first by $\frac{15}{2}$, the second by $\frac{15}{2}$ and the third by $\frac{105}{2}$. This means that we need to solve $3a + 5c + 15e = 0, 3b + 5d = 0, 15a + 21c + 35e = 0$. Here is the elimination:

$$\begin{pmatrix} 3 & 0 & 5 & 0 & 15 \\ 0 & 3 & 0 & 5 & 0 \\ 15 & 0 & 21 & 0 & 35 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 5 & 0 & 15 \\ 0 & 3 & 0 & 5 & 0 \\ 0 & 0 & -4 & 0 & -40 \end{pmatrix} \text{ and so}$$

$c = -10e, b = -\frac{5}{3}d, a = -\frac{5}{3}c - 5e = \frac{50}{3}e - 5e = \frac{35}{3}e$. This means that $p(x) = \frac{35}{3}ex^4 - \frac{5}{3}dx^3 - 10ex^2 + dx + e = \frac{e}{3}(35x^4 - 30x^2 + 3) - \frac{d}{3}(5x^3 - 3x)$ and so $p_3(x) = 35x^4 - 30x^2 + 3$ and $p_4(x) = 5x^3 - 3x$ constitute a basis of the orthogonal complement of $P^{(2)}$. This complement has dimension 2.
(c) We need to use Gram-Schmidt here. Set $q_3(x) = p_3(x) = 35x^4 - 30x^2 + 3$. Then
$\|q_3\|^2 = \langle q_3, q_3 \rangle = \int_{-1}^{1}(35x^4 - 30x^2 + 3)(35x^4 - 30x^2 + 3)dx = \frac{128}{9}$
$\langle q_3, p_4 \rangle = \int_{-1}^{1}(35x^4 - 30x^2 + 3)(5x^3 - 3x)dx = 0$.
This calculation shows that p_3 and p_4 are already an orthogonal basis.