# Math 415 - Assignment 9 Solutions

Problems: 6.2.2, 6.2.6, 7.1.2, 7.1.5, 7.1.14, 7.2.3, 7.2.4, 7.2.5, 7.2.25, 7.2.29

Problem 6.2.2

(a) Let's label the wires as follows: $(1) = 1 \to 2, (2) = 1 \to 3, (3) = 1 \to 4, (4) = 2 \to 3, (5) = 2 \to 4$. Then the incidence matrix is

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

(b) Given unit resistances, the conductance matrix $C$ is the identity. We also know that node 4 is grounded, so we can eliminate column 4 from $A$ to get the reduced matrix $A^*$ and we also eliminate the fourth entry of the external current vector $f$ to get the reduced vector $f^*$. Then the equilibrium system is governed by

$$K^* u^* = f^* \text{ where } K^* = (A^*)^T A^* \text{ and } A^* = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, f^* = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

(c) $$K^* = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} \text{ so}$$

$$u^* = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} & \frac{1}{2} \\ \frac{3}{8} & \frac{5}{8} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{15}{8} \\ \frac{9}{8} \\ \frac{3}{2} \end{pmatrix}$$

(d) The corresponding voltage vector is

$$v = Au = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{15}{8} \\ \frac{9}{8} \\ \frac{3}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{3}{8} \\ \frac{15}{8} \\ -\frac{3}{8} \\ \frac{9}{8} \end{pmatrix}$$

and therefore we should attach the bulb to wire 3.

Problem 6.2.6

First we need to label the nodes. Let the left node be 1, the right node be 2, the top node be 3 and the bottom node 4. Then label the wires as

$(1) = 1 \to 2, (2) = 1 \to 3, (3) = 1 \to 4, (4) = 2 \to 3, (5) = 2 \to 4, (6) = 3 \to 4$

Then the incidence and conductance matrices and battery vector are (battery in wire 1)

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, C = \tfrac{1}{3} I, b = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let's ground the last node and so form

$$A^* = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, K^* = \frac{1}{3} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}$$

$$f^* = -(A^*)^T Cb = -\frac{1}{3}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{10}{3} \\ \frac{10}{3} \\ 0 \end{pmatrix}$$

Then we need to solve $K^* u^* = f^*$, so

$$u^* = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\frac{10}{3} \\ \frac{10}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} -\frac{10}{3} \\ \frac{10}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix}$$

Now we find the current vector to be

$$y = Cv = C(A^* u^* + b) = \frac{1}{3}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ -\frac{5}{6} \\ -\frac{5}{6} \\ \frac{5}{6} \\ \frac{5}{6} \\ 0 \end{pmatrix}$$

The wire opposite the battery is wire 6, so the current in it is 0.

Problem 7.1.2

Generally speaking, if the answer to a subsection of this question is YES, then you need to find a matrix $A$ for which $F(\mathbf{x}) = A\mathbf{x}$

(a) Yes: $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(b) No: $F\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(c) No: there is a quadratic $xy$ term in one component

(d) Yes: $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(e) No: there are quadratic terms in each component

(f) Yes: $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Problem 7.1.5

(a) $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ -x \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(b) $F[e_1] = e_1, F[e_2] = \frac{1}{2}e_2 - \frac{\sqrt{3}}{2}e_3, F[e_3] = \frac{\sqrt{3}}{2}e_2 + \frac{1}{2}e_3$ so

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(c) $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(d) The rotation asked for transforms the standard axes back into themselves. If the rotation is counterclockwise, then $F[e_1] = e_2, F[e_2] = e_3, F[e_3] = e_1$ so

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(e) This part of the problem is tricky and involves some trigonometry. The rotation in this case takes $e_3$ and transforms it into a vector with one component in the $xy$ plane at a 45 degrees to both the $x$ and $y$ axes and a second component along the negative $z$ axis. A careful study of the angles involves gives $F[e_3] = \frac{2\sqrt{2}}{3}\left(\frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2\right) - \frac{1}{3}e_3 = \frac{2}{3}e_1 + \frac{2}{3}e_2 - \frac{1}{3}e_3$. Similarly $F[e_1] =$

$\frac{2\sqrt{2}}{3}\left(\frac{1}{\sqrt{2}}e_2+\frac{1}{\sqrt{2}}e_3\right)-\frac{1}{3}e_1=\frac{2}{3}e_2+\frac{2}{3}e_3-\frac{1}{3}e_1$ and $F[e_2]=\frac{2\sqrt{2}}{3}\left(\frac{1}{\sqrt{2}}e_1+\frac{1}{\sqrt{2}}e_3\right)-\frac{1}{3}e_2=\frac{2}{3}e_1+\frac{2}{3}e_3-\frac{1}{3}e_2$.
Thus

$$A=\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

(f) Here $F\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(g) First let us find a basis for the plane: $x=y-2z$ so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} y-2z \\ y \\ z \end{pmatrix}=y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}+z\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Let $w_1=(1,1,0)^T$ and $w_2=(-2,0,1)^T$. Nowdo Gram-Scmidt:
$v_1=w_1, ||v_1||^2=2, <w_2,v_1>=-2$,

$v_2=w_2-\frac{<w_2,v_1>}{||v_1||^2}v_1=\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}-\frac{-2}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}=\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, ||v_2||^2=3$. Then the orthogonal projection is

$$F[v]=F\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\frac{<v,v_1>}{||v_1||^2}v_1+\frac{<v,v_2>}{||v_2||^2}v_2=\frac{x+y}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}+\frac{-x+y+z}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{5}{6}x+\frac{1}{6}y-\frac{1}{3}z \\ \frac{1}{6}x+\frac{5}{6}y+\frac{1}{3}z \\ -\frac{1}{3}x+\frac{1}{3}y+\frac{1}{3}z \end{pmatrix}=\begin{pmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Problem 7.1.14

Set $e_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(a) Linearity: $L[aX+bY]=A(aX+bY)=aAX+bAY=aL[X]+bL[Y]$
Representation:

$$L[e_1]=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}=ae_1+0e_2+ce_3+0e_4$$

$$L[e_2]=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}=0e_1+ae_2+0e_3+ce_4$$

$$L[e_3]=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix}=be_1+0e_2+de_3+0e_4$$

$$L[e_4]=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}=0e_1+be_2+0e_3+de_4$$

Thus the matrix representation is $\mathbf{A}=\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$

(b) Linearity: $L[aX+bY]=(aX+bY)B=aXB+bYB=aL[X]+bL[Y]$
Representation:

$$L[e_1]=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}=pe_1+qe_2+0e_3+0e_4$$

$$L[e_2]=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}=re_1+se_2+0e_3+0e_4$$

$$L[e_3]=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ p & q \end{pmatrix}=0e_1+0e_2+pe_3+qe_4$$

$$L[e_4]=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ r & s \end{pmatrix}=0e_1+0e_2+re_3+se_4$$

Thus the matrix representation is $\mathbf{A} = \begin{pmatrix} p & r & 0 & 0 \\ q & s & 0 & 0 \\ 0 & 0 & p & r \\ 0 & 0 & q & s \end{pmatrix}$

(c) Linearity: $L[aX + bY] = A(aX + bY)B = aAXB + bAYB = aL[X] + bL[Y]$

Representation:

$L[e_1] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap & aq \\ cp & cq \end{pmatrix} = ape_1 + aqe_2 + cpe_3 + cqe_4$

$L[e_2] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ar & as \\ cr & cs \end{pmatrix} = are_1 + ase_2 + cre_3 + cse_4$

$L[e_3] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} bp & bq \\ dp & dq \end{pmatrix} = bpe_1 + bqe_2 + dpe_3 + dqe_4$

$L[e_4] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} br & bs \\ dr & ds \end{pmatrix} = bre_1 + bse_2 + dre_3 + dse_4$

Thus the matrix representation is $\mathbf{A} = \begin{pmatrix} ap & ar & bp & br \\ aq & as & bq & bs \\ cp & cr & dp & dr \\ cq & cs & dq & ds \end{pmatrix}$

Problem 7.2.3

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ be the matrix representing $L$. Then the matrix representing $L^2 = L \circ L$ is $A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and its action on $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is $L^2[v] = L^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ which is a 180° rotation. Since $\det A = 0 - (1)(-1) = 1$, $L$ itself is a rotation.

Problem 7.2.4

If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $L^2 = L \circ L = I$. This shows that $L$ undoes itself. Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ actually represents reflection through the line $x = y$, performing this reflection twice brings us back to the same point.

Problem 7.2.5

Here we have $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x - y \end{pmatrix}$ In this transformation the $x$ coordinate remains fixed and the $y$ coordinate is reflected along the vertical through the line $x = y$, that is $y$ goes to $x - (y - x) = 2x - y$. Thus $A^2 = I$.

Problem 7.2.25

(a) $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}^{-1} \begin{pmatrix} -3 & 2 & 2 \\ -3 & 1 & 3 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -3 & -1 & -2 \\ 6 & 1 & 6 \\ 1 & 1 & 0 \end{pmatrix}$

(a) $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -3 & 2 & 2 \\ -3 & 1 & 3 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(a) $B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -3 & 2 & 2 \\ -3 & 1 & 3 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & 0 & -\frac{12}{5} \\ 0 & -2 & 0 \\ -\frac{2}{5} & 0 & -\frac{1}{5} \end{pmatrix}$

Problem 7.2.29

(a) In the formula $B = S^{-1}AS$ the matrix $S$ is orthogonal because we know that its columns form

an orthonormal basis of $R^n$. Therefore we actually have $B = S^T AS$. Since $A$ is symmetric, this shows that
$B^T = (S^T AS)^T = S^T A^T S^{TT} = S^T AS = B$, so B is symmetric.

(b) In general no. Example: $\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} \\ -\frac{8}{3} & \frac{10}{3} \end{pmatrix}$