Math 415 Old Exam # 1

1. (12 points)

   (a) Find an LU factorization of the matrix
   $$A = \begin{pmatrix} 3 & 2 & 0 \\ 3 & 1 & -2 \\ -6 & -7 & -5 \end{pmatrix}$$

   (b) If a matrix B has the LU factorization
   $$B = \begin{pmatrix} 2 & 2 & -5 \\ -8 & -7 & 18 \\ 0 & 3 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$
   solve $Bx = \begin{pmatrix} -3 & 10 & -7 \end{pmatrix}^T$ without reducing the augmented matrix.

   (c) What is the value of $\det B$?

2. (12 points)

   (a) Give the definition of the inverse of a matrix A.

   (b) Write down a 3×3 permutation matrix P. Also write down its inverse.

   (c) Write down a 3×3 matrix E that performs an elementary row operation through matrix multiplication. Also write down its inverse.

   (d) Compute explicitly the inverse of the product PE.

   (e) What does the value of the determinant of a 3×5 matrix tell you about the existence of an inverse of that matrix?

3. (8 points)

   (a) Give a definition of the rank of a matrix A.

   (b) Find the rank of the matrix
   $$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

   (c) What is the rank of A^T ?

4. (12 points)

   (a) Give a definition of the concept of linear independence of a collection of vectors $\{v_1, v_2, \ldots, v_n\}$ from a vector space V.

   (b) For $V = M_{2\times 2}$ show that
   $v_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
   are linearly independent

5. (12 points)

   (a) What is meant by the span of a collection of vectors $\{v_1, v_2, \ldots, v_n\}$ from a vector space V?

   (b) Show that all vectors in the span of the two vectors
   $v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$
   lie in the plane in R^3 defined by $x + 2y + 3z = 0$.

   (c) What is the dimension of the subspace
   $W = \left\{ \begin{pmatrix} x & y & z \end{pmatrix}^T \text{ for which } x + 2y + 3z = 0 \right\}$ given the facts outlined in part b)? Be specific.

6. (10 points) The three vectors
   $v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
   form a basis for R^3 (you do NOT need to verify this!)

   (a) Express the vector $u = \begin{pmatrix} 5 & -8 & -1 \end{pmatrix}^T$ as a linear combination of these three vectors.

   (b) What are the "coordinates" of u relative to this basis?