Math 415 Old Exam # 1 and Solutions

1. (12 points)

   (a) Find an LU factorization of the matrix
   $$A = \begin{pmatrix} 3 & 2 & 0 \\ 3 & 1 & -2 \\ -6 & -7 & -5 \end{pmatrix}$$
   Answer: (5 points)
   $$A = \begin{pmatrix} 3 & 2 & 0 \\ 3 & 1 & -2 \\ -6 & -7 & -5 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 0 \\ 0 & -1 & -2 \\ 0 & -3 & -5 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = U$$
   $$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix} = L$$

   (b) If a matrix B has the LU factorization
   $$B = \begin{pmatrix} 2 & 2 & -5 \\ -8 & -7 & 18 \\ 0 & 3 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$
   solve $Bx = \begin{pmatrix} -3 & 10 & -7 \end{pmatrix}^T$ without reducing the augmented matrix.
   Answer: (6 points)
   $$b = \begin{pmatrix} -3 \\ 10 \\ -7 \end{pmatrix}$$
   Write $Bx = b$ as $LUx = b$ and this as $Ux = c$ where $Lc = b$. Solve first for c:
   $$\begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ -7 \end{pmatrix} \Rightarrow$$
   $c_1 = -3, c_2 = 10 + 4c_1 = -2, c_3 = -7 - 3c_2 = -1$. Now solve for x
   $$\begin{pmatrix} 2 & 2 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \Rightarrow$$
   $x_3 = 1, x_2 = -2 + 2x_3 = 0, x_1 = \frac{1}{2}(-3 - 2x_2 + 5x_3) = \frac{1}{2}(-3 - 0 + 5) = 1$.
   So
   $$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

   (c) What is the value of $\det B$?
   Answer: (1 point)
   -2, namely the product of the diagonal members of U.

2. (12 points)

   (a) Give the definition of the inverse of a matrix A.

Answer: (3 points)
The inverse of A is a matrix X for which $XA = AX = I$

(b) Write down a 3×3 permutation matrix P. Also write down its inverse.
Answer: (2 points)
Any matrix which permutes two rows of the identity will do, so try
$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Applying P again changes the rows back, so $P^{-1} = P$

(c) Write down a 3×3 matrix E that performs an elementary row operation through matrix multiplication. Also write down its inverse.
Answer: (3 points)
There are many choices. Here is one
$E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Its inverse is $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(d) Compute explicitly the inverse of the product PE.
Answer: (3 points)

$$(PE)^{-1} = E^{-1}P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) What does the value of the determinant of a 3×5 matrix tell you about the existence of an inverse of that matrix?
Answer: (1 points)
Nothing since the determinant and the inverse are only defined for square matrices

3. (8 points)

(a) Give a definition of the rank of a matrix A.
Answer: (2 points)
The rank of A is the number of pivots in the echelon form U of A.

(b) Find the rank of the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

Answer: (5 points)

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 8 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so the rank of A is 2

(c) What is the rank of A^T ?
Answer: (1 points)
rank A^T = rank A =2

4. (12 points)

(a) Give a definition of the concept of linear independence of a collection of vectors $\{v_1, v_2, \ldots, v_n\}$ from a vector space V.
Answer: (4 points)
$v_1, v_2, \ldots, v_n$ are linearly independent if the equation $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ implies that $c_1 = c_2 = \cdots = c_n = 0$

(b) For $V = M_{2\times 2}$ show that
$v_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
are linearly independent
Answer: (8 points)

$$c_1 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} c_1 + c_2 + c_3 & c_1 + c_2 \\ c_1 + c_3 & c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is equivalent to the equations $c_1 + c_2 + c_3 = 0, c_1 + c_2 = 0, c_1 + c_3 = 0, c_2 + c_3 = 0$ and these in turn are equivalent to the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So we need to row reduce the matrix on the left:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are three pivots, all the c_i's are zero and so we have linear independence.

5. (12 points)

(a) What is meant by the span of a collection of vectors $\{v_1, v_2, \ldots, v_n\}$ from a vector space V?
Answer: (2 points)
The span of a set is the set of all linear combinations of the vectors in the set

(b) Show that all vectors in the span of the two vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

lie in the plane in R^3 defined by $x + 2y + 3z = 0$.
Answer: (6 points)
The most general vector in the span is

$$a\begin{pmatrix}1\\1\\-1\end{pmatrix}+b\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}a+b\\a-2b\\-a+b\end{pmatrix}$$

Now verify that this vector satisfies the equation for the plane for all values of a and b:

$(a+b)+2(a-2b)+3(-a+b)=a+b+2a-4b-3a+3b=0$. Yes!

(c) What is the dimension of the subspace $W=\left\{\begin{pmatrix}x & y & z\end{pmatrix}^T \text{ for which } x+2y+3z=0\right\}$ given the facts outlined in part b)? Be specific.

Answer: (4 points)

If the two vectors are linearly independent, then they form a basis for the plane and so the dimension of the plane is 2. To test linear independence either note that neither vactor is a multiple of the other, or reduce the matrix with these as columns and show there are two pivots:

$$\begin{pmatrix}1&1\\1&-2\\-1&1\end{pmatrix}\to\begin{pmatrix}1&1\\0&-3\\0&2\end{pmatrix}\to\begin{pmatrix}1&1\\0&-3\\0&0\end{pmatrix}\text{ Done!}$$

6. (10 points) The three vectors

$$v_1=\begin{pmatrix}1\\1\\-1\end{pmatrix},\ v_2=\begin{pmatrix}1\\-2\\1\end{pmatrix}\text{ and } v_3=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

form a basis for R^3 (you do NOT need to verify this!)

(a) Express the vector $u=\begin{pmatrix}5 & -8 & -1\end{pmatrix}^T$ as a linear combination of these three vectors.

Answer: (8 points)

We need to find numbers x,y, and z such that

$$u=\begin{pmatrix}5\\-8\\-1\end{pmatrix}=x\begin{pmatrix}1\\1\\-1\end{pmatrix}+y\begin{pmatrix}1\\-2\\1\end{pmatrix}+z\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$=\begin{pmatrix}1&1&1\\1&-2&2\\-1&1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

By Gaussian elimination we have

$$\left(\begin{array}{ccc|c}1&1&1&5\\1&-2&2&-8\\-1&1&3&-1\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&5\\0&-3&1&-13\\0&2&4&4\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&5\\0&-3&1&-13\\0&0&\frac{14}{3}&-\frac{14}{3}\end{array}\right)$$

so $z=-1, y=-\frac{1}{3}(-13-z)=4, x=5-y-z=2$

(b) What are the "coordinates" of u relative to this basis?

Answer: (2 points)

$\begin{pmatrix}2 & 4 & -1\end{pmatrix}^T$