Math 415 Exam # 1 Solutions

1. (15 points) Let $LD_{2\times 2}$ denote the set of lower triangular 2×2 matrices, i.e. the set of matrices of the form

$$A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_3 \end{pmatrix}$$

   (a) Show that $LD_{2\times 2}$ is a subspace of $M_{2\times 2}$
   Solution: (5 points)

$$aA + bB = a\begin{pmatrix} x_1 & 0 \\ x_2 & x_3 \end{pmatrix} + b\begin{pmatrix} y_1 & 0 \\ y_2 & y_3 \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 & 0 \\ ax_2 + by_2 & ax_3 + by_3 \end{pmatrix}$$

   is lower triangular for any lower triangular matrices A and B and any scalars a and b so $LD_{2\times 2}$ is a subspace of $M_{2\times 2}$

   (b) Write down an equation that demonstrates that $LD_{2\times 2}$ is spanned by the vectors

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

   Solution: (4 points)

$$\begin{pmatrix} x_1 & 0 \\ x_2 & x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x_2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + x_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

   (c) Are v_1, v_2, and v_3 linearly independent? Justify your response.
   Solution: (4 points) By the last equation, the only way that we can have $c_1v_1 + c_2v_2 + c_3v_3 = 0$ is if

$$\begin{pmatrix} c_1 & 0 \\ c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   and so we see immediately that $c_1 = c_2 = c_3 = 0$.

   (d) What is the dimension of $LD_{2\times 2}$ and why?
   Solution: (2 points) By parts b and c the v_is form a basis of $LD_{2\times 2}$ and so the dimension is 3.

2. (15 points) Let $SLD_{3\times 3}$ denote the set of special lower triangular 3×3 matrices, i.e. the set of matrices of the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix}$$

   (a) Is $SLD_{3\times 3}$ a subspace of $M_{3\times 3}$? Explain.
   Solution: (2 points) No. It is not closed under scalar multiplication since a multiple of A will not have 1's along the diagonal.

(b) What is the permutted LU factorization of the elements of $SLD_{3\times 3}$, i.e. of the matrix A?
Solution: (3 points) A is already in factored form if we choose $P = I$, $L = A$ and $U = I$.

(c) What is the determinant of the elements of $SLD_{3\times 3}$, i.e. of the matrix A?
Solution: (2 points) $\det A$ is the product of the diagonal elements of $U = I$ and hence is 1.

(d) What is the inverse of

$$E_{12}(2) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution: (3 points)

$$E_{12}(2)^{-1} = E_{12}(-2) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) Compute the inverse of

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

Solution: (5 points) Since $B = E_{12}(2)E_{23}(3)$, we conclude that $B^{-1} = E_{23}(3)^{-1}E_{12}(2)^{-1} = E_{23}(-3)E_{12}(-2)$, that is

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 6 & -3 & 1 \end{pmatrix}$$

3. (10 points) What are the coordinates of the vector $p(x) = x^2 + 1$ in $P^{(2)}$ relative to the vectors

$$\begin{aligned} p_1(x) &= x^2 + x + 1 \\ p_2(x) &= 2x^2 + 3x - 2 \\ p_3(x) &= x^2 + 2x - 3 \end{aligned}$$

Explain and justify your answer carefully.
Solution: We must find constants c_1, c_2, c_3 such that

$$\begin{aligned} x^2 + 1 &= c_1(x^2 + x + 1) + c_2(2x^2 + 3x - 2) + c_3(x^2 + 2x - 3) \\ &= (c_1 + 2c_2 + c_3)x^2 + (c_1 + 3c_2 + 2c_3)x + (c_1 - 2c_2 - 3c_3) \end{aligned}$$

Comparing coefficients on both sides, we arrive at the equations

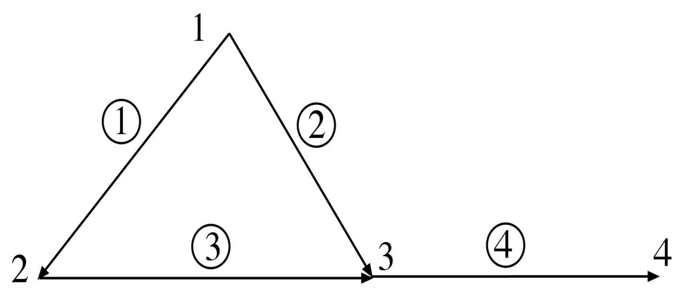
$$\begin{aligned} c_1 + 2c_2 + c_3 &= 1 \\ c_1 + 3c_2 + 2c_3 &= 0 \\ c_1 - 2c_2 - 3c_3 &= 1 \end{aligned}$$

Now use Gaussian elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & -2 & -3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -4 & -4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -4 \end{array}\right)$$

Since this system is incompatible, there are no coordinates. Hence p is not in the span of the other three vectors. These three vectors must be linearly deaendent, otherwise they would be a basis and they are not.

4. (10 points) Consider the following digraph:



(a) What is the incidence matrix A for this digraph?
Solution: (4 points)

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(b) What is a basis for $\ker A$?
Solution: (2 points) Since this digraph is connected, a basis for the kernel consists of the single vector with all 1's as entries.

(c) What is the dimension of $\operatorname{coker} A$ and why?
Solution: (4 points) The answer is 1. There are several ways to see this. A basis for the cokernel of A consists of the independent circuits and here there is by inspection exactly one circuit. Or, this follows from question 2.6.12 in the homework. Or it follows fron Euler's formula.

5. (10 points)

(a) Find a basis for rng$A$ for the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix}$$

Solution: (9 points) By Gaussian elimination

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & -1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Since the pivots appear in columns 1, 2, and 4, a basis for the range of A consists of columns 1, 2, and 4 of A, namely

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

(b) What is rank$A$?
Solution: (1 point) 3, the number of pivots.