Math 415 Old Exam # 2

1. (6 points) Let V be $C^0[0,1]$, the vector space of real-valued functions that are defined and continuous on $[0,1]$. Show that the subset

$$W = \{f \in V | f(0) = f(1)\}$$

is a subspace of V.

2. (12 points) A certain digraph has the incident matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

   (a) Draw a digraph that corresponds to this matrix.

   (b) What is a basis for $\ker A$?

   (c) By finding a basis for one of the other fundamental subspaces of A, find the independent circuits in this digraph.

3. (12 points) Let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y} \quad \text{where} \quad K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

   (a) Verify that $\langle \mathbf{x}, \mathbf{y} \rangle$ given above defines a valid inner product on R^3.

   (b) Find a basis for the set of all vectors that are orthogonal to $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^T$ in terms of the inner product given above.

4. (10 points) Find the least squares solution to the following system:

$$\begin{aligned} 2x + y &= 1 \\ x - y &= 2 \\ x + 5y &= 3 \end{aligned}$$

5. (10 points) Find all vectors in $P^{(3)}$ that are orthogonal to $p_1(x) = 1$ and $p_2(x) = x$ in the inner product

$$\langle p, q \rangle = \int_{-1}^{1} f(x)g(x)dx$$

6. (10 points)

   (a) An $m \times n$ matrix A has rank r. Which of the following statements is correct?

   i. A has r linearly independent columns
   ii. A has r linearly independent rows
   iii. Both i) and ii) are true
   iv. Neither i) or ii) is true
   Choice:

   (b) If the incident matrix for a connected digraph is $n \times n$, how many independent circuits are there in the digraph?

   i. 0
   ii. 1
   iii. more than 1
   iv. impossible to determine without further information
   Choice:

   (c) If an $m \times n$ matrix A has rank r, then the dimension of $\ker A$ is

   i. r
   ii. $n - r$
   iii. $m - r$
   iv. $m + n - 2r$
   Choice:

   (d) Here are two statements:

   Statement A : If x_1 and x_2 are both solutions of $Ax = 0$,
   then so is $c_1x_1 + c_1x_2$ for any real numbers c_1 and c_2
   Statement B : $\ker A$ is a subspace

   Which of the following options is the case:

   i. Neither statement implies the other
   ii. Statement A implies Statement B but B does not imply A
   iii. Statement B implies Statement A but A does not imply B
   iv. The two statements are equivalent
   Choice:

(e) In the representation $x = x^* + z$ of solutions of $Ax = b$,

i. $x - x^*$ satisfies the homogeneous system

ii. x^* is not unique if $\ker A$ is non-trivial

iii. both i) and ii) are true

iv. neither i) nor ii) is true

Choice: