Math 415 Old Exam # 2 and Solutions

1. (6 points) Let V be $C^0[0,1]$, the vector space of real-valued functions that are defined and continuous on $[0,1]$. Show that the subset

$$W = \{f \in V | f(0) = f(1)\}$$

is a subspace of V.
Answer: Let f and g lie in W. This means that f and g are continuous and $f(0) = f(1)$ and $g(0) = g(1)$. Then for any real numbers c and d

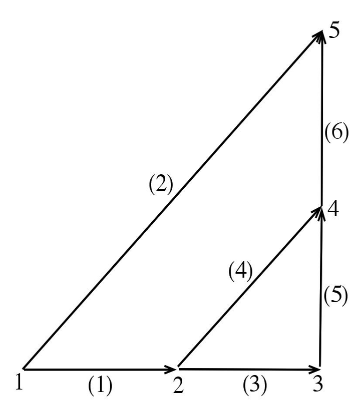
$$\begin{aligned}(cf + dg)(0) &= cf(0) + dg(0) \\ &= cf(1) + dg(1) \\ &= (cf + dg)(1)\end{aligned}$$

In addition, $cf + dg$ is also a continuous function and so $cf + dg$ lies once again in W. Thus W is a subspace of V.

2. (12 points) A certain digraph has the incident matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

   (a) Draw a digraph that corresponds to this matrix.
   Answer: (3 points)



(b) What is a basis for $\ker A$?

Answer: (1 points) The single vector $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$ is a basis for the kernel (proved in class).

(c) By finding a basis for one of the other fundamental subspaces of A, find the independent circuits in this digraph.

Answer: (8 points) In this case we need to find a basis of the cokernel of A:

$$A^T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If our variables are x, y, z, u, v, w, then $u = -v + w$, $z = v$, $y = -z - u = -v - (-v + w) = -w$, $x = -y = w$ so

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} w \\ -w \\ v \\ -v + w \\ v \\ w \end{pmatrix} = v \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Thus the independent circuits are (3)+(5)-(4) and (1)+(4)+(6)-(2).

3. (12 points) Let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y} \quad \text{where} \quad K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(a) Verify that $\langle \mathbf{x}, \mathbf{y} \rangle$ given above defines a valid inner product on R^3.

Answer: (6 points) In class we showed that all inner products on R^3 have the form $\mathbf{x}^T K \mathbf{y}$ provided that K is symmetric and positive definite. To show that K is positive definite use Gaussian Elimination:

$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

Since all pivots are positive, K is positive definite.

(b) Find a basis for the set of all vectors that are orthogonal to $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^T$ in terms of the inner product given above.

Answer: (6 points) Setting $\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T$, we need

$$0 = \left\langle \mathbf{x}, \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^T \right\rangle = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 2y$$

Thus $y = 0$ and x and z are free variables, so

$\mathbf{x} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and so e_1 and e_3 are a basis

4. (10 points) Find the least squares solution to the following system:

$$\begin{aligned} 2x + y &= 1 \\ x - y &= 2 \\ x + 5y &= 3 \end{aligned}$$

Answer: Set

$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Then we need to solve $Kx = f$ where

$K = A^T A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{pmatrix}^T \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 6 & 27 \end{pmatrix}$

$f = A^T b = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$. Thus

$\left(\begin{array}{cc|c} 6 & 6 & 7 \\ 6 & 27 & 14 \end{array}\right) \to \left(\begin{array}{cc|c} 6 & 6 & 7 \\ 0 & 21 & 7 \end{array}\right)$

Thus $y = \frac{1}{3}$ and $x = \frac{7}{6} - \frac{1}{3} = \frac{5}{6}$ and these are the coordinates of the least squares solution.

5. (10 points) Find all vectors in $P^{(3)}$ that are orthogonal to $p_1(x) = 1$ and $p_2(x) = x$ in the inner product

$$\langle p, q \rangle = \int_{-1}^{1} f(x) g(x) dx$$

Answer: Begin with a general vector from $P^{(3)}$, say $q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

Then we need

$$
\begin{aligned}
0 &= \int_{-1}^{1} 1(a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx \\
&= 2a_0 + 0a_1 + \frac{2}{3}a_2 + 0a_3 \\
0 &= \int_{-1}^{1} x(a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx \\
&= 0a_0 + \frac{2}{3}a_1 + 0a_2 + \frac{2}{5}a_3
\end{aligned}
$$

We conclude that a_2 and a_3 are free variables and $a_0 = -\frac{1}{3}a_2, a_1 = -\frac{3}{5}a_3$. So the general orthogonal vector is

$$
\begin{aligned}
q(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
&= a_2(x^2 - \frac{1}{3}) + a_3(x^3 - \frac{3}{5}x)
\end{aligned}
$$

6. (10 points)

(a) An $m \times n$ matrix A has rank r. Which of the following statements is correct?

i. A has r linearly independent columns
ii. A has r linearly independent rows
iii. Both i) and ii) are true
iv. Neither i) or ii) is true
Choice: iii

(b) If the incident matrix for a connected digraph is $n \times n$, how many independent circuits are there in the digraph?

i. 0
ii. 1
iii. more than 1
iv. impossible to determine without further information
Choice: ii

(c) If an $m \times n$ matrix A has rank r, then the dimension of $\ker A$ is

i. r
ii. $n - r$
iii. $m - r$
iv. $m + n - 2r$
Choice: ii

(d) Here are two statements:

Statement A : If x_1 and x_2 are both solutions of $Ax = 0$,
then so is $c_1 x_1 + c_1 x_2$ for any real numbers c_1 and c_2

Statement B : $\ker A$ is a subspace

Which of the following options is the case:

i. Neither statement implies the other
ii. Statement A implies Statement B but B does not imply A
iii. Statement B implies Statement A but A does not imply B
iv. The two statements are equivalent
Choice: iv

(e) In the representation $x = x^* + z$ of solutions of $Ax = b$,

i. $x - x^*$ satisfies the homogeneous system
ii. x^* is not unique if $\ker A$ is non-trivial
iii. both i) and ii) are true
iv. neither i) nor ii) is true
Choice: iii