Math 415 Exam # 2 Solutions

1. (10 points)

   (a) Demonstrate why
   $$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)x dx$$
   is not an inner product on $\mathcal{F}([-1,1])$.
   Solution: (5 points) Show that positivity fails. Take $f(x) = 1$, for example. Then
   $$\|1\|^2 = \langle 1, 1\rangle = \int_{-1}^{1} 1 \cdot 1 \cdot x dx = \frac{1}{2}x^2|_{-1}^{1} = 0$$
   but $f(x) = 1$ is not the zero vector.

   (b) Find a weighted inner product $\langle \mathbf{v}, \mathbf{w}\rangle = av_1w_1 + bv_2w_2 + cv_3w_3$ for which the vectors $\mathbf{v} = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}^T$ and $\mathbf{w} = \begin{pmatrix} 0 & -1 & 3 \end{pmatrix}^T$ are orthogonal.
   Solution: (5 points) The inner product of the two vectors is $\langle \mathbf{v}, \mathbf{w}\rangle = a \cdot 1 \cdot 0 + b \cdot 2 \cdot (-1) + c \cdot (-1) \cdot 3 = -2b - 3c$. For this to be zero, b and c would have to be of opposite signs. But for a, b, and c to define a weighted inner product, they all need to be positive. So this problem has no solution.

2. (15 points)

   (a) Find the $A = LDL^T$ factorization of
   $$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 9 \end{pmatrix}$$
   Solution: (6 points) By Gaussian-elimination
   $$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 6 \end{pmatrix}$$
   and so
   $$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

   (b) Given that
   $$K = \begin{pmatrix} 9 & 45 & 54 \\ 45 & 223 & 256 \\ 54 & 256 & 233 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 7 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 & 5 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}$$

write the quadratic form $q(x) = x^T Kx$ as a sum of squares of expressions in x. Is $q(x)$ positive definite, negative definite, or indefinite, and why?
Solution: (9 points) Set

$$y = L^T x = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 5x_2 + 6x_3 \\ x_2 + 7x_3 \\ x_3 \end{pmatrix}$$

Then $q(x) = x^T Kx = y^T Dy = 9y_1^2 - 2y_2^2 + 7y_3^2 = 9(x_1 + 5x_2 + 6x_3)^2 - 2(x_2 + 7x_3)^2 + 7x_3^2$. We see that $q(x)$ is indefinite because the diagonal elements of D are not all positive and not all negative.

3. (10 points)

   (a) Define what is meant by an orthonormal basis. Give an example of one in R^2 relative to the dot product that does NOT contain either of the standard basis vectors.
   Solution: (5 points) An orthonormal basis is a basis that is mutually orthogonal and in which all vectors are unit vectors. An example is

   $$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

   (b) Give a proof that if v_1, v_2, and v_3 are mutually orthogonal, then they are linearly independent.
   Solution: (5 points) If

   $$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

   then taking inner products of both sides with v_1 gives us

   $$0 = c_1 v_1 \cdot v_1 + c_2 v_2 \cdot v_1 + c_3 v_3 \cdot v_1 = c_1 ||v_1||^2 + c_2 0 + c_3 0 = c_1 ||v_1||^2$$

   Since v_1 is none-zero, we see that $c_1 = 0$. In the same way you can show that c_2 and c_3 are zero.

4. (10 points) Using distance measured by the dot product, find the point in the plane W spanned by v_1 and v_2 closest to the vector v where

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, v = \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix}$$

What is the distance between W and v?
Solution: You can either switch to an orthogonal basis and do an orthogonal projection or write down and solve the normal equations.
Method 1: Set

$$w_1 = v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, w_2 = v_2 - \frac{v_2 \cdot w_1}{\underset{2}{||w_1||^2}} w_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

Then the closest point is

$$w = \frac{v \cdot w_1}{||w_1||^2} w_1 + \frac{v \cdot w_2}{||w_2||^2} w_2 = \frac{12}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{3}{\frac{3}{2}} \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$$

The distance is the norm of $v - w$, namely $\sqrt{(6-5)^2 + (3-2)^2 + (6-7)^2} = \sqrt{3}$
Method 2: Set

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, K = A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix},$$

$$f = A^T v = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 21 \end{pmatrix}$$

and solve the normal equations:

$$Kx = f \Rightarrow x = K^{-1} f = \frac{1}{12 - 9} \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 12 \\ 21 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Then the closest vector is

$$w = 3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$$

5. (10 points)

   (a) State Fredholm's Criterion
   Solution: (3 points) $Ax = b$ has a solution if and only if b is orthogonal to the cokernal of A.

   (b) What restrictions must be placed on the right-hand side vector b of the system below to guarantee compatibility?

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}$$

   Solution: (4 points) If the coefficient matrix here is denoted A, then A^T is an incident matric for a digraph. A little inspection shows that node 1 connects to nodes 2, 3, and 4 and node 2 connects to 5, so the graph is connected. Thus the kernel of $A^T$is the vector of all 1's, and so the compatibility condition is

$$0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = b_1 + b_2 + b_3 + b_4 + b_5$$

(c) One solution of the system below is $x = \begin{pmatrix} 1 & 1 & -1 & 1 & 1 \end{pmatrix}^T$. What equations should be added to the system to find the unique solution of minimum norm?.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -2 \\ 0 \\ -2 \end{pmatrix}$$

Solution: (3 points) The coefficient matrix here is the incident matrix for a connected digraph, so its kernel is spanned by the vector of all ones. Thus, to find the solution of minimum norm we must add the equation

$$0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_1 + x_2 + x_3 + x_4 + x_5$$