Math 415 Old Exam # 3 and Solutions

1. (10 points)

   (a) The Gram-Schmidt procedure takes a collection of vectors v_1, v_2, v_3 and produces from them a set of orthogonal vectors w_1, w_2, w_3. Give formulas for the w_i's in terms of the v_i's.
   Solution: (3 points)

$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} w_1 \\ w_3 &= v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2 \end{aligned}$$

   (b) Given the polynomials $p_1(x) = 1, p_2(x) = x - 1, p_3(x) = x^2$ and the inner product $\langle p, q\rangle = \int_0^2 p(x)q(x)dx$, find the corresponding set of orthogonal polynomials given by the Gram-Schmidt procedure.
   Solution: (7 points) Set

$$q_1(x) = p_1(x) = 1$$

   Since

$$\|q_1\|^2 = \int_0^2 1dx = 2, \langle p_2, q_1\rangle = \int_0^2 (x-1)dx = \frac{2^2}{2} - 2 = 0$$

   we see that

$$q_2(x) = (x-1) - \frac{0}{2}1 = x - 1$$

   Now we compute

$$\begin{aligned} \|q_2\|^2 &= \int_0^2 (x-1)^2 dx = (x^2 - 2x + 1)dx = \frac{2^3}{3} - 2\frac{2^2}{2} + 2 = \frac{8}{3} - 2 = \frac{2}{3} \\ \langle p_3, q_1\rangle &= \int_0^2 x^2 dx = \frac{2^3}{3} = \frac{8}{3}, \langle p_3, q_2\rangle = \int_0^2 x^2(x-1)dx = \frac{2^4}{4} - \frac{2^3}{3} = 4 - \frac{8}{3} = \frac{4}{3} \end{aligned}$$

   So

$$q_3(x) = x^2 - \frac{8/3}{2}1 - \frac{4/3}{2/3}(x-1) = x^2 - \frac{4}{3} - 2x + 2 = x^2 - 2x + \frac{2}{3}$$

2. (10 points) Find the closest point to $b = \begin{pmatrix} 1 & 2 & -1 & 3 \end{pmatrix}^T$ in the subspace $W = \text{span}\left\{ \begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}^T, \begin{pmatrix} 1 & 1 & 0 & -1 \end{pmatrix}^T \right\}$
   Solution: First we need to find an othogonal basis of the subspace. Since

$\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & -1 \end{pmatrix} = 1 + 0 + 0 - 1 = 0$,we see that the basis is already orthogonal. Since

$$\begin{aligned} b \cdot \begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} = 2 \\ b \cdot \begin{pmatrix} 1 & 1 & 0 & -1 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & -1 \end{pmatrix} = 0 \\ \left\| \begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} \right\|^2 &= 6, \left\| \begin{pmatrix} 1 & 1 & 0 & -1 \end{pmatrix} \right\|^2 = 3 \end{aligned}$$

we see that the closest point is the projection

$$\frac{2}{6} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \frac{0}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

3. (8 points)

   (a) State Fredholm's Criterion for solutions of $Ax = b$
   Solution: (2 points) In order for $Ax = b$ to have a solution, b must be orthogonal to the cokernel of A

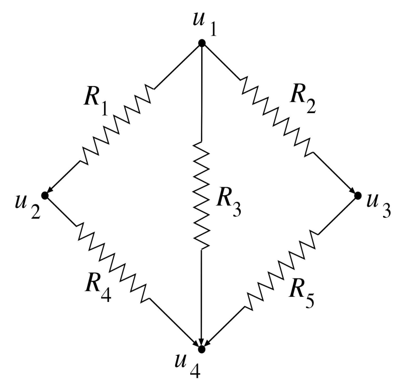
   (b) Given the following information:

   $$A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -2 & 1 \\ 1 & 3 & -5 & 2 \\ 5 & -1 & 9 & -6 \end{pmatrix}, \ker A = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}, \operatorname{co}\ker A = \left\{ \begin{pmatrix} 2 \\ 24 \\ -7 \\ 1 \end{pmatrix} \right\}$$

   find those restrictions on a vector b such that $Ax = b$ has a solution. Are these satisfied for $b = \begin{pmatrix} -1 & 1 & 4 & 6 \end{pmatrix}^T$?
   Solution: (3 points) We need $b \cdot \begin{pmatrix} 2 & 24 & -7 & 1 \end{pmatrix}^T = 2b_1 + 24b_2 - 7b_3 + b_4 = 0$. Yes:$-2 + 24 - 28 + 6 = 0$

   (c) For the matrix in part b), what additional restrictions on x are needed in order for the solution of $Ax = b$ to be the solution of minimum norm. Explain.
   Solution: (3 points) The solution of minimum norm is the unique solution orthogonal to the kernel, so just add the extra equation $x \cdot \begin{pmatrix} 1 & -1 & 0 & 1 \end{pmatrix}^T = x_1 - x_2 + x_4 = 0$

4. (10 points) For the electrical network shown here assume that an external current of size 1 amp enters node 1 and is removed from node 4. If all the wires have resistance 7 and node 2 is grounded, find the voltage potentials at each node. Explain your work.



Solution: The incident matrix for the network is

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Since node 2 is grounded, we eliminate column 2 of A and entry 2 of the external current vector f to get

$$A^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, f^* = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The conductance matrix here is $C = \frac{1}{7}I$. Now we form

$$K^* = A^{*T}CA^* = \frac{1}{7}\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

Finally we need to solve $K^*u = f^*$ for the voltage potentials. Multiply both sides by 7 before proceeding:

$$\left(\begin{array}{ccc|c} 3 & -1 & -1 & 7 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -7 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & -1 & -1 & 7 \\ 0 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ 0 & -\frac{4}{3} & \frac{8}{3} & -\frac{14}{3} \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & -1 & -1 & 7 \\ 0 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{5} & -\frac{14}{5} \end{array}\right)$$

Thus $u_4 = -\frac{14/5}{8/5} = -\frac{7}{4}, u_3 = \frac{3}{5}(\frac{7}{3} + \frac{4}{3}(-\frac{7}{4})) = 0, u_2 = 0$ (since it is grounded) and $u_1 = \frac{1}{3}(7 + 0 + (-\frac{7}{4})) = \frac{7}{4}$

5. (12 points)

(a) Explain why the function $L : R^3 \to R^3$ given by

$$L[v] = L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y - z \\ x + 2y + 2z \\ -x + y + 2z \end{pmatrix}$$

is linear. Be specific.

Solution: (4 points) It is enought to point out that the right-hand side above is a matrix times the vector $\begin{pmatrix} x & y & z \end{pmatrix}^T$ since we showed in class that linear functions from R^3 to R^3 all have this form. Alternately you must verify linearity directly:

$$\begin{aligned} L[av_1 + bv_2] &= \begin{pmatrix} 2(ax_1 + bx_2) + (ay_1 + by_2) - (az_1 + bz_2) \\ ax_1 + bx_2 + 2(ay_1 + by_2) + 2(az_1 + bz_2) \\ -ax_1 + bx_2 + (ay_1 + by_2) + 2(az_1 + bz_2) \end{pmatrix} \\ &= a\begin{pmatrix} 2x_1 + y_1 - z_1 \\ x_1 + 2y_1 + 2z_1 \\ -x_1 + y_1 + 2z_1 \end{pmatrix} + b\begin{pmatrix} 2x_2 + y_2 - z_2 \\ x_2 + 2y_2 + 2z_2 \\ -x_2 + y_2 + 2z_2 \end{pmatrix} \\ &= aL[v_1] + bL[v_2] \end{aligned}$$

(b) For the linear function

$$L[v] = L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x + 3y \\ 3x + 2y \end{pmatrix}$$

find its matrix representation relative to the satndard basis.

Solution: (2 points) This is the matrix

$$A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$$

(c) If the function L in part b) is invertible, find its inverse, expressing it in the same form as L

Solution: (6 points) Since $\det A = 8 - 9 = -1 \neq 0, A$ has an inverse and so L has an inverse. Its matrix representation is

$$A^{-1} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$$

and so

$$L^{-1}[v] = L^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + 3y \\ 3x - 4y \end{pmatrix}$$

6. (10 points) Find all eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -3 & 6 & -3 \\ -3 & 6 & -3 \end{pmatrix}$$

Solution:

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 0 \\ -3 & 6-\lambda & -3 \\ -3 & 6 & -3-\lambda \end{pmatrix} = -\lambda^2 (-3+\lambda)$$

so $\lambda = 0$ (double) and $\lambda = 3$
Case $\lambda = 3$:

$$(A - 3I)v = \begin{pmatrix} -3 & 0 & 0 \\ -3 & 3 & -3 \\ -3 & 6 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -3a \\ -3a + 3b - 3c \\ -3a + 6b - 6c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a = 0 \text{ and } b = c \text{ so choose } v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case $\lambda = 0$:

$$(A - 0I)v = \begin{pmatrix} 0 & 0 & 0 \\ -3 & 6 & -3 \\ -3 & 6 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -3a + 6b - 3c \\ -3a + 6b - 3c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a = 2b - c \text{ and so } v = \begin{pmatrix} 2b - c \\ b \\ c \end{pmatrix} = b \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

the last two vectors giving a basis for the $\lambda = 0$ eigenspace.