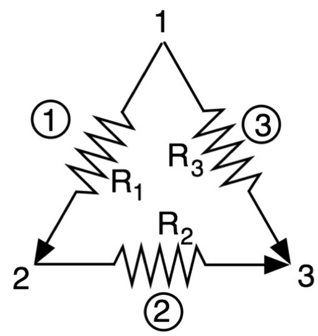
Math 415 Exam # 3 Solutions

1. (10 points) Given the circuit shown below, let the resistances in wires 1, 2, and 3 be $R_1 = \frac{1}{3}, R_2 = \frac{1}{6}, R_3 = \frac{1}{2}$ ohms, respectively. If 12 amps is input at node 1, 36 amps is input at node 2 and 48 amps is drawn out of the circuit at note 3, find the vector of voltage potentials if node 3 has been grounded.



Solutions: The incidence matrix for the circuit is

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

the conductance matrix is

$$C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$f = \begin{pmatrix} 12 \\ 36 \\ -48 \end{pmatrix}$$

With node 3 grounded we set

$$A^* = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then set

$$K^* = (A^*)^T C A^* = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}^T \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 9 \end{pmatrix}$$

and

$$f^* = \begin{pmatrix} 12 \\ 36 \end{pmatrix}$$

Then we just have to solve the normal equations $K^*u^* = f^*$ for u^*. We get

$$u^* = \begin{pmatrix} 5 & -3 \\ -3 & 9 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 36 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

and so

$$u = \begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix}$$

2. (10 points) Consider the vector space $P^{(2)}$. On this space consider the function (evaluated at vector $v(x) = ax^2 + bx + c$) defined as follows as a mapping from $P^{(2)}$ to $P^{(2)}$:

$$L[v] = L[ax^2 + bx + c] = bx^2 + cx + a$$

   (a) Verify that L is linear
   Solution: (5 points) Set $v_1 = a_1x^2 + b_1x + c_1$ and $v_2 = a_2x^2 + b_2x + c_2$. Then

$$\begin{aligned} L[cv_1 + dv_2] &= L[(ca_1 + da_2)x^2 + (cb_1 + db_2)x + (cc_1 + dc_2)] \\ &= (cb_1 + db_2)x^2 + (cc_1 + dc_2)x + (ca_1 + da_2) \\ &= c(b_1x^2 + c_1x + a_1) + d(b_1x^2 + c_1x + a_1) \\ &= cL[v_1] + dL[v_2] \end{aligned}$$

   (b) Relative to the basis $p_1(x) = 1, p_2(x) = x, p_3(x) = x^2$ in both domain and target spaces, what is the matrix A that represents L?
   Solution: (5 points)

$$\begin{aligned} L[p_1] &= L[1] = x = 0p_1(x) + 1p_2(x) + 0p_3(x) \\ L[p_2] &= L[x] = x^2 = 0p_1(x) + 0p_2(x) + 1p_3(x) \\ L[p_3] &= L[x^2] = 1 = 1p_1(x) + 0p_2(x) + 0p_3(x) \end{aligned}$$

   so

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

3. (10 points) Find a basis of R^3 relative to which the matrix representation of the following linear function is diagonal:

$$L[\mathbf{x}] = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} \mathbf{x}$$

Solution: We need to solve the eigenvalue problem for

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

So we compute

$$\det \begin{pmatrix} 3-\lambda & 2 & 1 \\ -1 & -\lambda & -1 \\ -2 & -2 & -\lambda \end{pmatrix} = 3\lambda^2 - \lambda^3 - 2\lambda = -\lambda(\lambda-1)(\lambda-2)$$

and see that λ is either 0 or 1 or 2. The corresponding eigenvectors are:
Case: $\lambda = 0:$

$$\begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow -a - c = 0 \text{ and } -2a - 2b = 0$$

so $c = -a, b = -a$, i.e. $\mathbf{v}_1 = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix}^T$
Case: $\lambda = 1:$

$$\begin{pmatrix} 2 & 2 & 1 \\ -1 & -1 & -1 \\ -2 & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow 2a + 2b + c = 0 \text{ and } a + b + c = 0$$

so $a + b = 0, b = -a$, $c = 0$, i.e. $\mathbf{v}_2 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T$
Case: $\lambda = 2:$

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow a + 2b + c = 0 \text{ and } a + b + c = 0$$

so $b = 0$ and $a + c = 0, c = -a$, i.e. $\mathbf{v}_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$
Since the eigenvalues are all distinct, $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent and so form a basis of R^3. Relative to this basis, the matrix representation of L is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

4. (10 points) Consider the matrix

$$A = \begin{pmatrix} p & q & 0 \\ 0 & p & r \\ 0 & 0 & p \end{pmatrix}, q \neq 0, r \neq 0$$

(a) Find the characteristic polynomial of A.
Solution:

$$p_A(\lambda) = \det \begin{pmatrix} p-\lambda & q & 0 \\ 0 & p-\lambda & r \\ 0 & 0 & p-\lambda \end{pmatrix} = (p-\lambda)^3$$

(b) Find all eigenvalues of A and their multiplicities.
Solution: The only eigenvalue of A is $\lambda = p$ and its multiplicity is 3

(c) Find bases of the eigenspaces of A and the dimensions of those eigenspaces.
Solution: We need to solve

$$\begin{pmatrix} p-p & q & 0 \\ 0 & p-p & r \\ 0 & 0 & p-p \end{pmatrix} x = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and this shows that $qb = 0, rc = 0$. Since q and r are non-zero, we conclude that $b = c = 0$, so we get the single eigenvector basis $\left\{ \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T \right\}$. The dimension of this eigenspace is 1.

(d) Define what is meant by a matrix A being incomplete.
Solution: A matrix is incomplete if for at least one of its eigenvalues the dimension of the corresponding eigenspace is different from the multiplicity of that eigenvalue.

(e) Find a basis of R^3 that includes the basis vectors for the eigenspaces of A.
Solution: The standard basis in R^3 will suffice.

5. (10 points) Find the matrix representation (relative to the standard basis) of the linear function $L : R^3 \to R^3$ that performs a reflection in the xy plane followed by a counter-clockwise rotation through 45° about the z axis.
Solution: A reflection in the xy plane is given by

$$M\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

and its matrix representation is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

A counter-clockwise rotation through 45° about the z axis has matrix representation

$$C = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix representation that we need is

$$A = CB = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$