Math 415 Final Examination - May 5, 2008 - R. Muncaster

Name: ______________________________

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Total	

Instructions:
a) You have 3 hours to complete the exam.
b) Include all calculations and justifications of your work.
Final answers are generally worth only a small fraction of the total mark. Remember that your goal is to demonstrate an understanding of the concepts and skills in the course.
c) Calculators are not permitted.
d) Good luck!

1. (10 points)

   (a) Define what it means for vectors $v_1, \ldots, v_n$ to be linearly independent. Be precise.

   (b) Prove that if the vectors $v_1, \ldots, v_n$ are mutually orthogonal, then they are also linearly independent.

2. (10 points) Let $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix}$

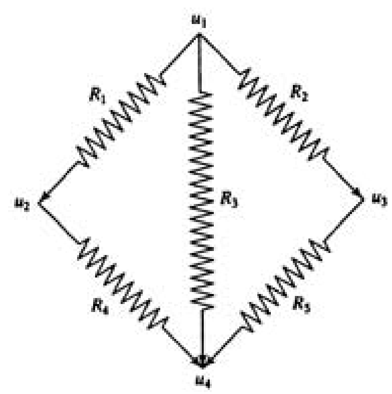
   (a) Find a basis for the cokernel of A. What is the rank of A?

(b) Find all values of the constant a for which the system

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a \\ 3 \\ 4 \end{pmatrix}$$

has a solution and what is the dimension of the solution set?

3. (15 points) Consider the electical network shown below



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Wires 1, 2, 4 and 5 have resistance 1 ohm and wire 3 has resistance 2 ohms and there is a 10 volt battery in wire 1. Find the general form of the voltage potentials at all nodes (i.e. do not assume that one of the nodes is grounded).

4. (10 points) Using the inner product

$$\langle x, y\rangle = x^T K y \text{ where } K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

turn the vectors

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, w_1 = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}, w_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

into an orthonormal set.

5. (15 points) Solve the initial value problem

$$\begin{aligned} \dot{x} &= 3x + y - z, \qquad x(0) = 1 \\ \dot{y} &= x + 3y - z, \qquad y(0) = 2 \\ \dot{z} &= 3x + 3y - z, \qquad z(0) = -1 \end{aligned}$$

(Hint: one eigenvalue is $\lambda = 1$ and another is $\lambda = 2$, so you just need to find the third one.).

6. (15 points)

(a) If a matrix A has an LDL^T factorization, does this matrix **have** to be symmetric? Explain.

(b) Is the $A = LDL^T$ factorization of a symmetric matrix A the same (with $Q = L$ and $D = \Lambda$) as the diagonalization $A = Q\Lambda Q^T$? Explain.

(c) Find the $Q\Lambda Q^T$ factorization of the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

7. (10 points) Find the closest point on the hyperplane $x + y + z + w = 0$ to the vector $b = \begin{pmatrix} 3 & 1 & 2 & 1 \end{pmatrix}^T$.

8. (10 points) Consider the vector space $P^{(2)}$. On this space consider the function (with $v(x) = ax^2 + bx + c$)

$$L[v] = L[ax^2 + bx + c] = bx^2 + cx + a$$

(a) Verify that L is linear

(b) Relative to the basis $p_1(x) = 1, p_2(x) = x, p_3(x) = x^2$, what is the matrix A that represents L?