Math 415 Final Examination and Solutions - May 5, 2008 - R. Muncaster

1. (10 points)

   (a) Define what it means for vectors $v_1, \ldots, v_n$ to be linearly independent. Be precise.
   Solution: (4 points) Vectors $v_1, \ldots, v_n$ are linearly independent if the only constants $c_1, \ldots, c_n$ for which

   $$c_1 v_1 + \cdots + c_n v_n = 0$$

   are $c_1 = 0, \ldots, c_n = 0$.

   (b) Prove that if the vectors $v_1, \ldots, v_n$ are mutually orthogonal, then they are also linearly independent.
   Solution: (6 points) Let

   $$c_1 v_1 + \cdots + c_n v_n = 0$$

   Take the inner product of both sides with v_1 to get

   $$\begin{aligned} c_1 v_1 \cdot v_1 + c_2 v_2 \cdot v_1 + \cdots + c_n v_n \cdot v_1 &= 0 \\ \text{so } c_1 \|v_1\| + c_2 0 + \cdots + c_n 0 &= 0 \\ \text{or } c_1 \|v_1\| &= 0 \end{aligned}$$

   Assuming all the vectors are non-zero (one bonus mark for mentioning this), we conclude that $c_1 = 0$. In the same way we can show that all the c's are zero

2. (10 points) Let

   $$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix}$$

   (a) Find a basis for the cokernel of A. What is the rank of A?
   Solution: (points 6)

   $$A^T = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & -1 & 7 \\ 1 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \\ 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   so $y = z, x = -y - 2z = -3z$. Thus

   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

   and so $\begin{pmatrix} -3 & 1 & 1 \end{pmatrix}^T$ forms a basis of the cokernel.

(b) Find all values of the constant a for which the system

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a \\ 3 \\ 4 \end{pmatrix}$$

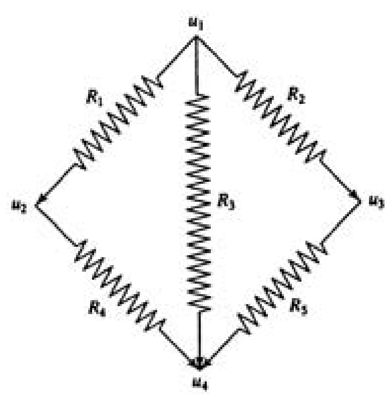
has a solution and what is the dimension of the solution set?
Solution: (4 points) By Fredholm's Criterion, this system will have a solution if the right-hand side is orthogonal to the cokernel, i.e. if

$$\begin{pmatrix} -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & 3 & 4 \end{pmatrix} = -3a + 3 + 4 = 0, \text{ i.e. } a = \frac{7}{3}$$

The calculation in part a) shows that A has two linearly independent rows, i.e. rank A is 2, so there are two linearly independent columns too, i.e. 2 pivots. This means there are two free variables and so the solution set is 2 dimensional.

3. (15 points) Consider the electical network shown below. Wires 1, 2, 4 and 5 have resistance 1 ohm and wire 3 has resistance 2 ohms and there is a 10 volt battery in wire 1. Find the general form of the voltage potentials at all nodes (i.e. do not assume that one of the nodes is grounded).
Solution: The incident matrix, conductance matrix, and battery vector here are

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



Then

$$\begin{aligned}
K &= A^T C A \\
&= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{5}{2} & -1 & -1 & -\frac{1}{2} \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -\frac{1}{2} & -1 & -1 & \frac{5}{2} \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
f &= -A^T C b \\
&= -\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ 10 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

Now we need to solve $Ku = f$:

$$\left(\begin{array}{cccc|c} 4 & -1 & -1 & -2 & -10 \\ -1 & 2 & 0 & -1 & 10 \\ -1 & 0 & 2 & -1 & 0 \\ -2 & -1 & -1 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} \frac{5}{2} & -1 & -1 & -\frac{1}{2} & -10 \\ 0 & \frac{8}{5} & -\frac{2}{5} & -\frac{6}{5} & 6 \\ 0 & -\frac{2}{5} & \frac{8}{5} & -\frac{6}{5} & -4 \\ 0 & -\frac{6}{5} & -\frac{6}{5} & \frac{12}{5} & -2 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|c} \frac{5}{2} & -1 & -1 & -\frac{1}{2} & -10 \\ 0 & \frac{8}{5} & -\frac{2}{5} & -\frac{6}{5} & 6 \\ 0 & 0 & \frac{3}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & -\frac{3}{2} & \frac{3}{2} & \frac{5}{2} \end{array}\right) \to \left(\begin{array}{cccc|c} \frac{5}{2} & -1 & -1 & -\frac{1}{2} & -10 \\ 0 & \frac{8}{5} & -\frac{2}{5} & -\frac{6}{5} & 6 \\ 0 & 0 & \frac{3}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So $u_3 = u_4 - \frac{5}{3}$, $u_2 = \frac{5}{8}(6 + \frac{2}{5}(u_4 - \frac{5}{3}) + \frac{6}{5}u_4) = \frac{10}{3} + u_4$, $u_1 = \frac{2}{5}(-10 + \frac{10}{3} + u_4 + u_4 - \frac{5}{3} + \frac{1}{2}u_4) = -\frac{10}{3} + u_4$, i.e.

$$u = \begin{pmatrix} -\frac{10}{3} + u_4 \\ \frac{10}{3} + u_4 \\ u_4 - \frac{5}{3} \\ u_4 \end{pmatrix} = \begin{pmatrix} -\frac{10}{3} \\ \frac{10}{3} \\ -\frac{5}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

where $t = u_4$.

4. (10 points) Using the inner product

$$\langle x, y \rangle = x^T K y \text{ where } K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

turn the vectors

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, w_2 = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

into an orthonormal set.
Solution: First set $v_1 = w_1$. Then we compute

$$\|v_1\|^2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^T \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 12$$

$$\langle v_1, w_2 \rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^T \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} = 0$$

Thus we can take $v_2 = w_2$ since v_1 and w_2 are already orthogonal in this inner product. Now for some more calculations:

$$\|v_2\|^2 = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}^T \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} = 42$$

$$\langle v_1, w_3 \rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^T \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = -4$$

$$\langle v_2, w_3 \rangle = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}^T \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 29$$

Thus we define v_3 via Gram-Schmidt:

$$\begin{aligned} v_3 &= w_3 - \frac{\langle v_1, w_3 \rangle}{\|v_1\|^2} v_1 - \frac{\langle v_2, w_3 \rangle}{\|v_2\|^2} v_2 \\ &= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - \frac{-4}{12} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{29}{42} \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} \\ \frac{3}{14} \\ 0 \end{pmatrix} \end{aligned}$$

and note that

$$\|v_3\|^2 = \begin{pmatrix} -\frac{3}{7} \\ \frac{3}{14} \\ 0 \end{pmatrix}^T \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -\frac{3}{7} \\ \frac{3}{14} \\ 0 \end{pmatrix} = \frac{9}{14}$$

Now all we need to do is normalize:

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{12}} \\ \frac{2}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \end{pmatrix}, u_2 = \begin{pmatrix} \frac{4}{\sqrt{42}} \\ \frac{5}{\sqrt{42}} \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} -\frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ 0 \end{pmatrix}$$

5. (15 points) Solve the initial value problem

$$\begin{aligned} \dot{x} &= 3x + y - z, \qquad x(0) = 1 \\ \dot{y} &= x + 3y - z, \qquad y(0) = 2 \\ \dot{z} &= 3x + 3y - z, \qquad z(0) = -1 \end{aligned}$$

(Hint: one eigenvalue is $\lambda = 1$ and another is $\lambda = 2$, so you just need to find the third one.).
Solution: We need to solve the eigenvalue problem first:

$$\begin{aligned} \det \begin{pmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{pmatrix} &= -8\lambda + 5\lambda^2 - \lambda^3 + 4 \\ &= -(\lambda - 1)(\lambda - 2)^2 \end{aligned}$$

so $\lambda = 1$ and $\lambda = 2$ (double).
Case $\lambda = 1$:

$$(A - I)v = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 3 & 3 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a + b - c \\ a + 2b - c \\ 3a + 3b - 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Subtract the first two to get $a - b = 0$, so $b = a$. Then the first gives $c = 2a + b = 3a$. Thus we can use $v = \begin{pmatrix} 1 & 1 & 3 \end{pmatrix}^T$
Case $\lambda = 2$:

$$(A - 2I)v = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a + b - c \\ a + b - c \\ 3a + 3b - 3c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $c = a + b$ so

$$v = \begin{pmatrix} a \\ b \\ a + b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The general solution is then

$$\begin{aligned} x(t) &= c_1 e^t \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t + c_2 e^{2t} \\ c_1 e^t + c_3 e^{2t} \\ 3c_1 e^t + c_2 e^{2t} + c_3 e^{2t} \end{pmatrix} \end{aligned}$$

The initial condition then gives us

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = x(0) = \begin{pmatrix} c_1 + c_2 \\ c_1 + c_3 \\ 3c_1 + c_2 + c_3 \end{pmatrix}$$

Thus $c_2 = 1 - c_1, c_3 = 2 - c_1, -1 = 3c_1 + (1 - c_1) + (2 - c_1) = c_1 + 3$. We then have $c_1 = -4, c_2 = 5, c_3 = 6$ and

$$x(t) = \begin{pmatrix} -4e^t + 5e^{2t} \\ -4e^t + 6e^{2t} \\ -12e^t + 5e^{2t} + 6e^{2t} \end{pmatrix}$$

6. (15 points)

(a) If a matrix A has an LDL^T factorization, does this matrix **have** to be symmetric? Explain.
Solution: (3 points) D is diadonal and hence symmetric, so $A^T = (LDL^T)^T = (L^T)^T D^T L^T = LDL^T = A$, so yes.

(b) Is the $A = LDL^T$ factorization of a symmetric matrix A the same (with $Q = L$ and $D = \Lambda$) as the diagonalization $A = Q\Lambda Q^T$? Explain.
Solution: (2 points) No since L is lower triangular and Q is orthogonal.

(c) Find the $Q\Lambda Q^T$ factorization of the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Solution: (10 points) We need to solve the eigenvalue problem here:

$$\det \begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix} = -9\lambda + 6\lambda^2 - \lambda^3 = -\lambda(-3+\lambda)^2$$

Case $\lambda = 0$:

$$(A - 0I)v = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a - b - c \\ -a + 2b - c \\ -a - b + 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Subtract the first two to get $3a - 3b = 0$, so $b = a$. Then the first gives $c = 2a - b = a$. Thus we can use $v = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$
Case $\lambda = 3$:

$$(A - 3I)v = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a - b - c \\ -a - b - c \\ -a - b - c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $c = -a - b$ so

$$v = \begin{pmatrix} a \\ b \\ -a - b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

These two are both orthogonal to the first eigenvector, but are not orthogonal to each other, so we need a little Gram-Schmidt:

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$$

gives an alternate to the third vector orthogonal to the second. Now just normaliz to get

$$Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

7. (10 points) Find the closest point on the hyperplane $x + y + z + w = 0$ to the vector $b = \begin{pmatrix} 3 & 1 & 2 & 1 \end{pmatrix}^T$.
Solution: First find a basis for the hyperplane: $x = -y - z - w$ so

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -y-z-w \\ y \\ z \\ w \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now we set

$$A = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, K = A^T A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, f = A^T b = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$$

and proceed to solve the normal equations $Kx = j$:

$$\begin{pmatrix} 2 & 1 & 1 & -2 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & -2 \\ 0 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & -2 \\ 0 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{4}{3} & -1 \end{pmatrix}$$

So $x_3 = -\frac{3}{4}, x_2 = -\frac{1}{3}x_3 = \frac{1}{4}, x_1 = \frac{1}{2}(-2 - (-\frac{3}{4}) - (\frac{1}{4})) = -\frac{3}{4}$ and therefore the closest point is

$$-\frac{3}{4}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5 \\ -3 \\ 1 \\ -3 \end{pmatrix}$$

8. (10 points) Consider the vector space $P^{(2)}$. On this space consider the function (with $v(x) = ax^2 + bx + c$)

$$L[v] = L[ax^2 + bx + c] = bx^2 + cx + a$$

(a) Verify that L is linear

Solution: (5 points) Set $v_1 = a_1 x^2 + b_1 x + c_1$ and $v_2 = a_2 x^2 + b_2 x + c_2$. Then

$$\begin{aligned} L[cv_1 + dv_2] &= L[(ca_1 + da_2)x^2 + (cb_1 + db_2)x + (cc_1 + dc_2)] \\ &= (cb_1 + db_2)x^2 + (cc_1 + dc_2)x + (ca_1 + da_2) \\ &= c(b_1 x^2 + c_1 x + a_1) + d(b_1 x^2 + c_1 x + a_1) \\ &= cL[v_1] + dL[v_2] \end{aligned}$$

(b) Relative to the basis $p_1(x) = 1, p_2(x) = x, p_3(x) = x^2$, what is the matrix A that represents L?

Solution: (5 points)

$$\begin{aligned} L[p_1] &= L[1] = x = 0p_1(x) + 1p_2(x) + 0p_3(x) \\ L[p_2] &= L[x] = x^2 = 0p_1(x) + 0p_2(x) + 1p_3(x) \\ L[p_3] &= L[x^2] = 1 = 1p_1(x) + 0p_2(x) + 0p_3(x) \end{aligned}$$

so

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$