Math 415 - Supplementary Handout 1
Olver & Shakiban - Sections 1.3, 1.4

Let us denote by $E_{ij}(a)$ the elementary matrix that represents adding a times row i to row j. Thus, in Equation (1.19) of Section 1.3,

$$\begin{aligned} E_1 &= E_{12}(-2) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = E_{13}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ \text{and} \quad E_3 &= E_{23}(\frac{1}{2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \end{aligned}$$

The advantage of the notation $E_{ij}(a)$ is that it tells us immediately what row operation is being performed by this elementary matrix. For example, the matrix L_1 is the matrix that "undoes" E_1. Therefore L_1 should add 2 times row 1 to row 2, and therefore

$$L_1 = E_{12}(2) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly

$$L_2 = E_{13}(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L_3 = E_{23}(-\frac{1}{2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

We know from our book that E_1 and L_1 are inverses, E_2 and L_2 are inverses, and E_3 and L_3 are inverses: each undoes the other. A neat way of saying this in general is that

$$E_{ij}(a) \text{ and } E_{ij}(-a) \text{ are inverses, i.e. } E_{ij}(a)E_{ij}(-a) = I \text{ and } E_{ij}(-a)E_{ij}(a) = I$$

Here is an important fact about elementary matrices. While in general matrices do not commute, any two elementary matrices $E_{ij}(a)$ and $E_{ik}(b)$ with the same first index commute. Indeed, for 3x3 matrices

$$\begin{aligned} E_{12}(a)E_{13}(b) &= \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \\ E_{13}(b)E_{12}(a) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \end{aligned}$$

Each product represents the same thing: add a time row 1 to row 2 *and* add b times row 1 to row 3. This will be an important fact shortly.

In terms of this new notation, Equation (1.20) can be written

$$E_{23}(\frac{1}{2})E_{13}(-1)E_{12}(-2)A = U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{5}{2} \end{pmatrix}$$

and therefore

$$A = E_{12}(2)E_{13}(1)E_{23}(-\frac{1}{2})U = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \end{pmatrix} U$$

Now let us consider permutation matrices as introduced in Section 1.4. There are special permutation matrices that just interchange two rows. Denote by P_{ij} the permutation matrix that represents interchanging rows i and j For example, for 4x4 matrices (applying this same operation to the identity matrix)

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A general permutation matrix is a product of (potentially) several of these special permutation matrices and so represents (potentially) more than one interchange or rows.

Given this new notation, let us look in detail at Example 1.12 where a *permuted LV* decomposition is calculated. We begin with

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & -1 \\ -3 & -5 & 6 & 1 \\ -1 & 2 & 8 & -2 \end{pmatrix}$$

As in the text we add multiples of row 1 to all other rows to create 0's below the pivot, but this time we write the calculation as the equation

$$E_{14}(1)E_{13}(3)E_{12}(-2)A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 4 & 7 & -2 \end{pmatrix}$$

We see now that the next pivot location (2,2) contains zero, so we need to pivot (i.e interchange some rows so the the pivot is valid, that is, non-zero). We therefore interchange rows 2 and 3 and write the result as

$$P_{23}E_{14}(1)E_{13}(3)E_{12}(-2)A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 4 & 7 & -2 \end{pmatrix}$$

Next we create all 0's below the pivot at (2,2) by applying another row operation:

$$E_{24}(-4)P_{23}E_{14}(1)E_{13}(3)E_{12}(-2)A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -5 & -6 \end{pmatrix}$$

Now we see that the pivot at (3,3) contains zero and so we must pivot again. Interchange rows 3 and 4 and write the result as

$$P_{34}E_{24}(-4)P_{23}E_{14}(1)E_{13}(3)E_{12}(-2)A = U = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

First we note that

$$\begin{aligned} P_{23}E_{14}(1)E_{13}(3)E_{12}(-2) &= E_{14}(1)P_{23}E_{13}(3)E_{12}(-2) \\ &= E_{14}(1)E_{12}(3)E_{13}(-2)P_{23} \end{aligned}$$

In the first line we have used the fact that adding a multiple of line 1 to line 4 has no effect on interchanging lines 2 and 3 (i.e. the corresponding matrix operations commute). In the second line we note that if we had first interchanged rows 2 and row 3, then we would achieve the same reult by thereafter adding -2 times row 1 to row 3 and then adding 3 times row 1 to row 2. Now we note the following:

$$P_{34}E_{24}(-4) = E_{23}(-4)P_{34}$$

This says that first adding -4 times row 2 to row 4 and then interchanging rows 3 and 4 is the same as first interchanging rows 3 and 4 and after that adding -4 times row 2 to row 3. By similar reasoning, we see that

$$P_{34}E_{14}(1)E_{12}(3)E_{13}(-2) = E_{13}(1)E_{12}(3)E_{14}(-2)P_{34}$$

i.e. interchange the roles of rows 3 and 4 on the right-hand side. Finally we use the fact that the matrices E_{12}, E_{13}, and E_{14} commute to write (1) as

$$\begin{aligned} E_{23}(-4)P_{34}E_{14}(1)E_{12}(3)E_{13}(-2)P_{23}A &= E_{23}(-4)E_{13}(1)E_{12}(3)E_{14}(-2)P_{34}P_{23}A \\ &= E_{23}(-4)E_{14}(-2)E_{13}(1)E_{12}(3)P_{34}P_{23}A \\ &= U \end{aligned}$$

Applying inverses, we conclude that

$$P_{34}P_{23}A = E_{12}(-3)E_{13}(-1)E_{14}(2)E_{23}(4)U$$

that is

$$PA = LU$$

where

$$P = P_{34}P_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$L = E_{12}(-3)E_{13}(-1)E_{14}(2)E_{23}(4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

These are the same results as appear in Example 1.12.