Math 415 - Supplementary Handout 3

Construction of the Matrix for a CounterClockwise Rotation About a General Axis

If the axis of a rotation in R^3 is a general vector

$$\mathbf{z} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and the rotation is counterclockwise through an angle θ, what matrix represents this rotation? Here is a derivation

First let us find an orthogonal basis for R^3 that includes the vector $\mathbf{z}$. One vector that is clearly orthogonal to $\mathbf{z}$ is

$$\mathbf{z}_1 = \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$$

Another one is clearly

$$\mathbf{w} = \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix}$$

but this is not orthogonal to $\mathbf{z}_1$ in general. Let's use Gram-Schmidt to construct from $\mathbf{w}$ a vector $\mathbf{z}_2$ that is orthogonal to both $\mathbf{z}$ and $\mathbf{z}_1$:

$$\begin{aligned}
\mathbf{z}_2 &= \mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{z}}{\|\mathbf{z}\|^2}\mathbf{z} - \frac{\mathbf{w} \cdot \mathbf{z}_1}{\|\mathbf{z}_1\|^2}\mathbf{z}_1 \\
&= \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} - \frac{0}{\|\mathbf{z}\|^2}\mathbf{z} - \frac{bc}{a^2+b^2}\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} \\
&= \frac{a}{a^2+b^2}\begin{pmatrix} ac \\ bc \\ -a^2-b^2 \end{pmatrix}
\end{aligned}$$

The fraction out front of this vector can be dropped since that will not affect orthogonality of the three, so let's set

$$\mathbf{z}_2 = \begin{pmatrix} ac \\ bc \\ -a^2-b^2 \end{pmatrix}$$

In order to talk correctly about "counterclockwise" we need to make sure our basis vectors are ordered correctly. Let use the cross product to determine the correct order. We have

$$\mathbf{z} \times \mathbf{z}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} = \begin{pmatrix} ca \\ cb \\ -a^2-b^2 \end{pmatrix} = \mathbf{z}_2$$

so we take the basis vectors in the order $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}$ so that the first vector plays the role of the x-axis, the second corresponds to the y-axis and $\mathbf{z}$ corresponds to the z-axis in this alternate basis. We conclude that relative to this special orthogonal basis, the matrix representing our rotation is

$$B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To finish the problem we just have to switch back to the standard basis. If the matrix representing our rotation relative to the standard basis is A and we set $S = (\mathbf{z}_1, \mathbf{z}_2, \mathbf{z})$, then Eqn (7.28) of the text says that $B = S^{-1}AS$. This means that $A = SBS^{-1}$. Before we compute this, it would be useful to replace $\mathbf{z}_1$, $\mathbf{z}_2$,and $\mathbf{z}$ by corresponding unit vectors since then S would be orthogonal and so easy to invert. This gives us

$$S = \begin{pmatrix} \frac{b}{\sqrt{a^2+b^2}} & \frac{ac}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}} & \frac{a}{\sqrt{a^2+b^2+c^2}} \\ \frac{-a}{\sqrt{a^2+b^2}} & \frac{bc}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}} & \frac{b}{\sqrt{a^2+b^2+c^2}} \\ 0 & \frac{-a^2-b^2}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}} & \frac{c}{\sqrt{a^2+b^2+c^2}} \end{pmatrix}$$

To simplify calculations, let's assume that the original vector $\mathbf{z}$ is a unit vector. Then S simplifies to

$$S = \begin{pmatrix} \frac{b}{\sqrt{a^2+b^2}} & \frac{ac}{\sqrt{a^2+b^2}} & a \\ \frac{-a}{\sqrt{a^2+b^2}} & \frac{bc}{\sqrt{a^2+b^2}} & b \\ 0 & \frac{-a^2-b^2}{\sqrt{a^2+b^2}} & c \end{pmatrix}$$

We then find that

$$\begin{aligned} A &= SBS^{-1} = SBS^T \\ &= \begin{pmatrix} (b^2+c^2)\cos\theta + a^2 & -ab\cos\theta - c\sin\theta + ab & -ac\cos\theta + b\sin\theta + ac \\ -ab\cos\theta + c\sin\theta + ab & (a^2+c^2)\cos\theta + b^2 & -bc\cos\theta - a\sin\theta + bc \\ -ac\cos\theta - b\sin\theta + ac & -bc\cos\theta + a\sin\theta + bc & (a^2+b^2)\cos\theta + c^2 \end{pmatrix} \end{aligned}$$

As an example, if we choose $\mathbf{z} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T$ and rotate through 180°, then

$$\begin{aligned} A &= \begin{pmatrix} \frac{2}{3}\cos\pi + \frac{1}{3} & -\frac{1}{3}\cos\pi - \frac{1}{\sqrt{3}}\sin\pi + \frac{1}{3} & -\frac{1}{3}\cos\pi + \frac{1}{\sqrt{3}}\sin\pi + \frac{1}{3} \\ -\frac{1}{3}\cos\pi + \frac{1}{\sqrt{3}}\sin\pi + \frac{1}{3} & \frac{2}{3}\cos\pi + \frac{1}{3} & -\frac{1}{3}\cos\pi - \frac{1}{\sqrt{3}}\sin\pi + \frac{1}{3} \\ -\frac{1}{3}\cos\pi - \frac{1}{\sqrt{3}}\sin\pi + \frac{1}{3} & -\frac{1}{3}\cos\pi + \frac{1}{\sqrt{3}}\sin\pi + \frac{1}{3} & \frac{2}{3}\cos\pi + \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \end{aligned}$$

which is the matrix representation asked for in Problem 7.1.5 (e).