Math 415 - Supplementary Handout 4
Construction of the Matrix for a Reflection Through a General Plane Through the Origin

Given a plane $ax + by + cz = 0$ in R^3 passing through the origin, what matrix represents a reflection of vectors through this plane? Here is a derivation:

The normal to the plane is the vector

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We need to take a general vector $\mathbf{x}$ and decompose it into its projection onto the normal – call this $\mathbf{x}_n$ – and its projection onto the plane – call this $\mathbf{x}_p$. Then $\mathbf{x} = \mathbf{x}_n + \mathbf{x}_p$ and the reflected vector will be $\mathbf{y} = -\mathbf{x}_n + \mathbf{x}_p$

First let us find an orthogonal basis for R^3 that includes the vector $\mathbf{n}$. One vector that is clearly orthogonal to $\mathbf{n}$ is

$$\mathbf{n}_1 = \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$$

Another one is clearly

$$\mathbf{w} = \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix}$$

but this is not orthogonal to $\mathbf{n}_1$ in general. Let's use Gram-Schmidt to construct from $\mathbf{w}$ a vector $\mathbf{n}_2$ that is orthogonal to both $\mathbf{n}$ and $\mathbf{n}_1$:

$$\begin{aligned} \mathbf{n}_2 &= \mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} - \frac{\mathbf{w} \cdot \mathbf{n}_1}{\|\mathbf{n}_1\|^2}\mathbf{n}_1 \\ &= \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} - \frac{0}{\|\mathbf{n}\|^2}\mathbf{n} - \frac{bc}{a^2 + b^2}\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} \\ &= \frac{a}{a^2 + b^2}\begin{pmatrix} ac \\ bc \\ -a^2 - b^2 \end{pmatrix} \end{aligned}$$

The fraction out front of this vector can be dropped since that will not affect orthogonality of the three, so let's set

$$\mathbf{n}_2 = \begin{pmatrix} ac \\ bc \\ -a^2 - b^2 \end{pmatrix}$$

If

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

then the projection of $\mathbf{x}$ on the plane is

$$
\begin{aligned}
\mathbf{x}_p &= \frac{\mathbf{x}\cdot\mathbf{n}_1}{\|\mathbf{n}_1\|^2}\mathbf{n}_1 + \frac{\mathbf{x}\cdot\mathbf{n}_2}{\|\mathbf{n}_2\|^2}\mathbf{n}_2 \\
&= \frac{bx-ay}{a^2+b^2}\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} + \frac{acx+bcy-(a^2+b^2)z}{(a^2+b^2)(a^2+b^2+c^2)}\begin{pmatrix} ac \\ bc \\ -a^2-b^2 \end{pmatrix} \\
&= \frac{1}{a^2+b^2+c^2}\begin{pmatrix} xb^2-ayb+xc^2-acz \\ -bax-bcz+c^2y+a^2y \\ -acx-bcy+za^2+zb^2 \end{pmatrix}
\end{aligned}
$$

The projection onto the normal is then

$$
\begin{aligned}
\mathbf{x}_n &= \mathbf{x}-\mathbf{x}_p \\
&= \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{1}{a^2+b^2+c^2}\begin{pmatrix} xb^2-ayb+xc^2-acz \\ -bax-bcz+c^2y+a^2y \\ -acx-bcy+za^2+zb^2 \end{pmatrix} \\
&= \frac{1}{a^2+b^2+c^2}\begin{pmatrix} xa^2+ayb+acz \\ yb^2+bax+bcz \\ zc^2+acx+bcy \end{pmatrix}
\end{aligned}
$$

The reflected vector is then

$$
\begin{aligned}
\mathbf{y} &= -\mathbf{x}_n+\mathbf{x}_p \\
&= \frac{1}{a^2+b^2+c^2}\left(-\begin{pmatrix} xa^2+ayb+acz \\ yb^2+bax+bcz \\ zc^2+acx+bcy \end{pmatrix} + \begin{pmatrix} xb^2-ayb+xc^2-acz \\ -bax-bcz+c^2y+a^2y \\ -acx-bcy+za^2+zb^2 \end{pmatrix}\right) \\
&= \frac{1}{a^2+b^2+c^2}\begin{pmatrix} -xa^2-2ayb-2acz+xb^2+xc^2 \\ -yb^2-2bax-2bcz+c^2y+a^2y \\ -zc^2-2acx-2bcy+za^2+zb^2 \end{pmatrix} \\
&= A\mathbf{x}
\end{aligned}
$$

where A is the matrix representation of the reflection, namely

$$
A = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & -b^2+a^2+c^2 & -2bc \\ -2ac & -2bc & -c^2+a^2+b^2 \end{pmatrix}
$$

As an example, if we choose $\mathbf{n} = (0,0,1)^T$ then

$$
\begin{aligned}
A\mathbf{x} &= \begin{pmatrix} -0+0+1 & -2(0)(0) & -2(0)(1) \\ -2(0)(0) & -0+0+1 & -2(0)(1) \\ -2(0)(1) & -2(0)(1) & -1+0+0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}
\end{aligned}
$$

a reflection through the xy plane.