Math 415
Lecture 1
Sec 1.1. Solutions of Linear Systems

$$\begin{aligned} x + 2y + z &= 2 \\ 2x + 6y + z &= 7 \\ x + y + 4z &= 3 \end{aligned}$$

What operations can we perform to this system without changing its' sol'n set and yet make it simpler?

Linear Sys Op'n 1: Add a multiple of one eq'n to another

Eqn 2 + $(-2)\times$ Eqn 1 $\rightarrow$ new Eqn 2
Eqn 3 + $(-1)\times$ Eqn 1 $\rightarrow$ new Eqn 3

$$\begin{aligned} x + 2y + z &= 2 \\ 2y - z &= 3 \\ -y + 3z &= 1 \end{aligned}$$

Eqn 3 + $(\frac{1}{2})\times$ Eqn 2 $\longrightarrow$ new Eq'n 3

$$\begin{aligned} x + 2y + z &= 2 \\ 2y - z &= 3 \\ \tfrac{5}{2}z &= \tfrac{5}{2} \rightarrow z = 1 \end{aligned}$$

Triangular

$2y - 1 = 3 \rightarrow y = 2$

$x + 2(2) + 1 = 2 \rightarrow x = -3$

$\underbrace{\qquad\qquad}_{\text{Back Substitution}}$

The trick here is that we want to do this more efficiently, without, say, writing dow x, y, z every time!
What other operations don't change the solution set?

① Mult an eq'n by a constant (non-zero)
② Interchange the eq'ns.

Sec 1.2 So we write the r.h.s. and the coeffs of x, y, z as arrays of numbers:

vector $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{matrix} (x_i) \\ \text{(column vector)} \end{matrix}$ (can have row vectors too!)

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = (b_j)$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

m eq'ns in n unknowns

$$A = \text{matrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

(coefficient matrix of our system)

$m \times n$ matrix

no. of rows no. of columns

$$= (a_{ij})$$

$$\underbrace{\text{columns of } A}_{} = (a_1 \cdots a_n)$$

Vectors are $1 \times n$ matrices!

Two Fundamental Operations:

Addition - $A+B = (a_{ij}) + (b_{ij}) \equiv (a_{ij}+b_{ij})$

(A & B must be same size) add corresp entries

Scalar Multiplication - $cA = c(a_{ij}) \equiv (c\,a_{ij})$, $c \in \mathbb{R}$.

multip each entry by c

Extra Operation: Matrix multiplication

If $A = (a_{ij})$ is $m \times n$ and $B = (b_{ij})$ is $n \times p$, then $C = AB = (c_{ij})$ is $m \times p$ and c_{ij} is defined by

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}.$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 0 & 2 \\ -1 & 1 \end{pmatrix} = \cdots$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \cdots$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \cdots$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \cdots, \quad \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \cdots$$

generally matrix mult'n does not commute!

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + z \\ 2x + 6y + z \\ x + y + 4z \end{pmatrix} = b = \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

ie $A\vec{x} = b$

Important observation:

$$AB = A\,(b_1 \cdots b_p) = (Ab_1 \;\cdots\; Ab_p)$$

Convince yousel!

$b_1 \cdots b_p$: columns of B

$Ab_1 \cdots Ab_p$: columns of AB

Special Matrices:

<u>Identity</u>

$$I = I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad n \times n$$

It has the property that

$$AI = A \text{ for all } A$$
$$IA = A \text{ for all } A$$

If A is $m \times n$, the first I is I_n and the second I is I_m.

Check it for yourself in simple cases!

Thus I is the multiplicative identity.

<u>Square Matrices</u> Matrices that are $n \times n$.

<u>Diagonal Matrices</u>

These are square matrices where all off-diagonal entries are zero

We can write them as

diag$(d_1, \dots, d_n)$. So, for example, $I_n = \text{diag}(\underbrace{1, 1, \dots, 1}_{n})$

<u>Upper Triangular</u> Square matrices where all entries below the diagonal are zero. Lower triangular is define in the same way

<u>Special upper triangular</u> Upper triangular with 1's on the diagonal.