Math 415

Lecture 10

Sec 2.4 Bases & Dimension

Def'n: A basis of a vector space V is a set of elements of V that

a) spans V

b) is linearly indep.

In $\mathbb{R}^n$ the standard basis is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

i.e. $x_1 e_1 + \cdots + x_n e_n = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ so spans, and this is 0 iff $x_1 = \cdots = x_n = 0$.

Thm Every basis of V has the same number of elements. This number is called the dimension of V (see pg 102)

Example $P^{(n)}$ has dimension $n+1$

Take $e_0(x) = 1,\ e_1(x) = x,\ e_2(x) = x^2, \ldots, e_n(x) = x^n$.

Then $p(x) = a_n x^n + \cdots + a_1 x + a_0 = a_n e_n(x) + \cdots + a_1 e_1(x) + a_0 e_0(x)$

How about linear independence?

$$c_1 \cdot 1 + c_2 \cdot x + c_3 x^2 + \cdots + c_{n+1} x^n = 0$$

set $x=0 \Rightarrow c_1 = 0$. Now divide out an x to get

$$c_2 + c_3 x + c_4 x^2 + \cdots + c_{n+1} x^{n-1} = 0$$

Set $x=0 \Rightarrow c_2 = 0$. Keep going until all c_i's are zero.

Or use successive differentiations!

Lemma If $v_1, \dots, v_n$ span V and $w_1, \dots, w_k$ lie in V and $k > n$, then $w_1, \dots, w_k$ are linearly dependent.

Pf Write $\quad w_j = a_{1j} v_1 + a_{2j} v_2 + \cdots + a_{nj} v_n \quad , j = 1, \dots, k$

Then test for linear dependence:

$$0 = \sum_{j=1}^{k} c_j w_j = \sum_{j=1}^{k} c_j \sum_{i=1}^{n} a_{ij} v_i$$

$$= \sum_{i=1}^{n} \Big(\sum_{j=1}^{k} a_{ij} c_j \Big) v_i$$

Choose the c_j's so that $AC = 0$ and $c \neq 0$. Always possible since homog system with more eqns than unknowns (always free vars)

Pf of Thm Let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_k\}$ be bases where $k > n$. Then $v_1, \dots, v_n$ span V, so $w_1, \dots, w_k$ are linearly dependent - contradicts the properties of a basis.

Conclusions: $\mathbb{R}^n$ has dimension n. $P^{(n)}$ has dimension $n+1$. $T^{(2)}$ has dimension 5 (finish this!)

Theorem Let $\{v_1, \dots, v_n\}$ be a basis of V and let

$$x = c_1 v_1 + \cdots + c_n v_n$$

be a representation of $x \in V$ as a linear combo of basis vectors. Then this representation is unique. The c_i are called the components of x relative to $\{v_1, \dots, v_n\}$.

Pf Suppose $x = \sum_{i=1}^{n} c_i v_i$ and $x = \sum_{i=1}^{n} b_i v_i$. Then

$$0 = x - x = \sum_{i=1}^{n} c_i v_i - \sum_{i=1}^{n} b_i v_i = \sum_{i=1}^{n} (c_i - b_i) v_i$$

combo that is zero

By linear independence $c_i - b_i = 0$ for all i, i.e. uniqueness.

For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + \cdots + x_n e_n \Rightarrow x_i$ are components of x relative to $\{e_1, \dots, e_n\}$.

What's involved in finding coords relative to a different basis?

$$x = c_1 v_1 + \cdots + c_n v_n = \underbrace{(v_1 \cdots v_n)}_{A} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{c} = Ac$$

$\Rightarrow$ solve a linear system with x on the r.h.s.

Talk about the wavelet basis problem.

EXAMPLE 2.35

A Wavelet Basis. The vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \tag{2.21}$$

form a basis of $\mathbb{R}^4$. This is verified by performing Gaussian Elimination on the corresponding 4×4 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

to check that it is nonsingular. This is a very simple example of a *wavelet basis.* Wavelets play an increasingly central role in modern signal and digital image processing, [16, 66].

How do we find the coordinates of a vector, say $\mathbf{x} = \left(4, -2, 1, 5\right)^T$, relative to the wavelet basis? We need to find the coefficients c_1, c_2, c_3, c_4 so that

$$\mathbf{x} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + c_3\,\mathbf{v}_3 + c_4\,\mathbf{v}_4.$$

We use (2.13) to rewrite this equation in matrix form $\mathbf{x} = A\,\mathbf{c}$, where

$$\mathbf{c} = (c_1, c_2, c_3, c_4)^T .$$

Solving the resulting linear system by Gaussian Elimination produces $c_1 = 2$, $c_2 = -1$, $c_3 = 3$, $c_4 = -2$, which are the coordinates of

$$\mathbf{x} = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 5 \end{pmatrix} = 2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 - 2\mathbf{v}_4$$

$$= 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

in the wavelet basis.