Math 415
Lecture 11

Sec 2.5 Four Fundamental Subspaces

Let A be an $m \times n$ matrix, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and consider

$$Ax = b$$

Def'n The kernel of A, denoted $\ker A$, (also null space of A) is

$$\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

We use the linear property of A:

$$A(c_1 x + c_2 y) = c_1 Ax + c_2 Ay$$

Prop'n $\ker A$ is a subspace of $\mathbb{R}^n$. (Use linearity)

Def'n The range of A, denoted $\operatorname{rng} A$, (also image of A or column space of A) is

$$\operatorname{rng} A = \{Ax \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

Prop'n $\operatorname{rng} A$ is a subspace of $\mathbb{R}^m$. Let $v \in \operatorname{rng} A$, $w \in \operatorname{rng} A$. Then there are vectors $x, y \in \mathbb{R}^n$ s.t. $v = Ax$, $w = Ay$. Then

$$v + w = Ax + Ay = A(x+y), \quad cv = cAx = A(cx) \checkmark$$

How do we compute these?

Kernel of A:

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{pmatrix} \to \cdots \to \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\Rightarrow \quad x = 2z + 17w, \quad y = z + 10w$$

$$\Rightarrow \quad x = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = z \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 17 \\ 10 \\ 0 \\ 1 \end{pmatrix}$$

so these span $\ker A$
and form a basis (show lin indep)

Superposition Principles

a) If $z_1, \dots, z_k$ are solutions of $Az = 0$ (homog lin sys) then so is $c_1 z_1 + \cdots + c_k z_k$ for any scalars $c_1, \dots, c_k$. (True since this is exactly the same as saying that $\ker A$ is a subspace)

b) Let $Ax_1^* = b_1$, $Ax_2^* = b_2$. Then

$$A(c_1 x_1^* + c_2 x_2^*) = c_1 b_1 + c_2 b_2$$

c) The general solution of $Ax = b$ has the form
$$x = x^* + z$$

where x^* is any one sol'n of $Ax^* = b$ and $z \in \ker A$.

<u>Pf</u> $Ax = b$, $Ax^* = b \Rightarrow A(x - x^*) = Ax - Ax^* = b - b = 0$

ie $z = x - x^* \in \ker A$.

<u>Example</u>

$$\left.\begin{array}{lcl} x - z = 3 & \longrightarrow & x = 3 + z \\ y - 2z = 1 & \longrightarrow & y = 1 + 2z \end{array}\right\}$$

$$x - 2y + 3z = 1 \longrightarrow 3 + z - 2(1 + 2z) + 3z = 1$$

$$3 + z - 2 - 4z + 3z = 1$$

$$1 = 1 \checkmark$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+z \\ 1+2z \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}}_{\substack{x^* \\ \text{a particular} \\ \text{sol'n}}} + z \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{\ker A = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\}}$$

We conclude that the following are equivalent (A is $m \times n$)

i) $\ker A = \{0\}$

ii) $\operatorname{rank} A = n$

iii) $Ax = b$ has no free variables

iv) $Ax = b$ has a unique sol'n for each $b \in \operatorname{rng} A$.

Def'n Then <u>corange</u> of A, denoted $\text{corng}\, A$, is
$$\text{corng}\, A = \text{rng}\, A^T \quad (= \text{span of the rows of } A)$$

The cokernel of A, denoted $\text{coker}\, A$, is

$$\text{coker}\, A = \ker A^T = \text{set of solutions of } \underbrace{A^T x = 0}_{\text{called the adjoint homog system}}$$

Theorem Let A be $m \times n$ and have rank r. Then
$$\dim \text{corng}\, A = \dim \text{rng}\, A = \text{rank}\, A = \text{rank}\, A^T = r$$
$$\dim \ker A = n - r, \quad \dim \text{coker} = m - r.$$