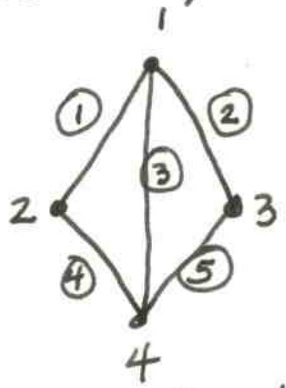
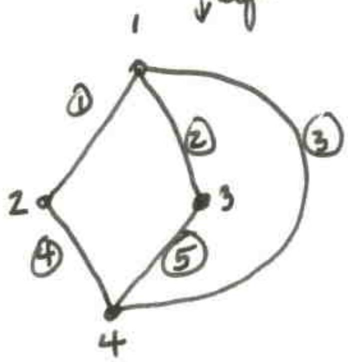
Math 415
Lecture 12

Sec 2.6 Graphs and Incident Matrices



equiv.



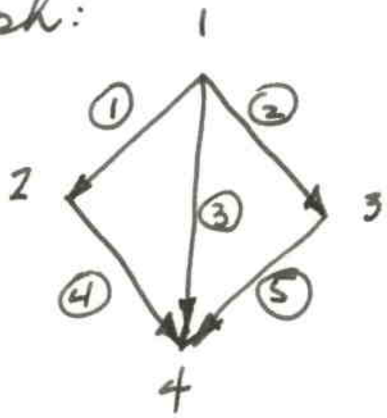
Graph: an object with vertices $v_1, \dots, v_k$ and edges $e_1, \dots, e_n$. Two vertices are connected by at most one arc or edge

CONNECTED!

Path: An ordered list of edges $e_1, \dots, e_p$ where e_{i+1} begins from the vertex where e_i ends and no edge is used more than once.

Circuit: A path beginning and ending at the same vertex. (and we don't care about the vertex)

Digraph:



Incident Matrix:

$$\begin{array}{c|c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} & \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = A \end{array}$$

begins at

ends at

Notice that $Az = 0$ where A is any incidence matrix and $z = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ (Why?).

<u>Prop'n</u> If A is the incidence matrix for a connected digraph, then $\ker A = \operatorname{span}\left\{\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\right\}$ is one-dimensional.

<u>Pf</u> Suppose $Az = 0$. If e_k joins v_i to v_j, then the k^{th} eq'n here is $z_i = z_j$. By induction, if there is a path from v_i to v_j then $z_i = z_j$. By connectedness, all components of z are the same. Done.

Let's look at the cokernel of A:

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2 free vars $\to$ $\dim \operatorname{coker} A = 2$

$$\operatorname{Span}\left\{ \underset{y_1}{\begin{pmatrix}1\\0\\-1\\1\\0\end{pmatrix}}, \underset{y_2}{\begin{pmatrix}0\\1\\-1\\0\\1\end{pmatrix}} \right\}$$

These corresp to special circuits. They are viewed as "independent" since they have at least one edge not in common with the other.

What is $-y_1$?

What is $y_1 - y_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$? Note the cancelation along edge 3.

The dimension of coker A is called the number of independent circuits.

But we already know that $\dim \ker A = n - r = 1$,

so $\operatorname{rank} A = r = n-1$.

Then

$$
\begin{aligned}
&\#\text{vertices} + \#\text{indep circuits} \\
&= n + m - r \\
&= n + m - (n-1) \\
&= m + 1 \\
&= \#\text{edges} + 1 \quad \leftarrow \text{Euler's Formula.}
\end{aligned}
$$

Talk about planar graphs and "holes".