Math 415 Lecture 13

Section 3.1 Inner Products

Dot product: $\langle v, w \rangle = v \cdot w = v_1 w_1 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i$
$= v^T w$

$$\| v \| = \sqrt{\langle v, v \rangle} = \sqrt{v_1^2 + \cdots + v_n^2}$$

Def'n An inner product is a map from $V \times V$ to $\mathbb{R}$ satisfying:

Bilinearity a) $\langle cu + dv, w \rangle = c \langle u, w \rangle + d \langle v, w \rangle$
$\langle u, cv + dw \rangle = c \langle u, v \rangle + d \langle u, w \rangle$ $\Big\} \forall c, d \in \mathbb{R}$

Symmetry b) $\langle u, v \rangle = \langle v, u \rangle$

Positivity c) $\langle v, v \rangle > 0$ for all $v \neq 0$, $\langle 0, 0 \rangle = 0$

The norm corresponding to $\langle \, , \rangle$ is
$\| v \| = \sqrt{\langle v, v \rangle}$.

Examples: In $\mathbb{R}^2$, define $\langle v, w \rangle = 2 v_1 w_1 + 5 v_2 w_2$
"weighted Euclidean".

$\| v \|^2 = \langle v, v \rangle = 2 v_1^2 + 5 v_2^2 > 0$ unless $v_1 = v_2 = 0$.
Certainly linear in v and in w. And symmetric.
Why would this norm and inner product be useful?

Example $\langle v, w \rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + 4 v_2 w_2$
$\| v \|^2 = \langle v, v \rangle = v_1^2 - 2 v_1 v_2 + 4 v_2^2 = (v_1 - v_2)^2 + 3 v_2^2 \geq 0$
etc.

Function Spaces:

$C^0[a,b]$ = space of continuous real valued f'ns on $[a,b]$.

$$\langle f, g \rangle = \int_a^b f(x)\,g(x)\,dx.$$

Check linearity and symmetry.

$$\| f \| = \sqrt{\int_a^b f^2(x)\,dx}$$

when is this 0?

Example: On $[a,b] = [0, \frac{\pi}{2}]$ we have

$$\langle \sin x, \cos x \rangle = \int_0^{\pi/2} \sin x \cos x \, dx = \frac{1}{2} \sin^2 x \Big|_0^{\pi/2}$$

$$= \frac{1}{2}$$

$$\| \sin x \| = \sqrt{\int_0^{\pi/2} \sin^2 x \, dx} = \sqrt{\frac{\pi}{4}}$$

$$\| 1 \| = \sqrt{\int_0^{\pi/2} 1^2 \, dx} = \sqrt{\frac{\pi}{2}}.$$