# Math 415 Lecture 14

## 3.2 Inequalities

Cauchy-Schwartz Inequality.

$$|\langle v,w\rangle| \le \|v\|\,\|w\| \quad \text{for all } v,w\in V$$

Pf. If $w=0$, both sides are $0$, so done.
Assume $w\neq 0$. Then

$$0 \le \|v+tw\|^2 = \langle v+tw, v+tw\rangle = \langle v,v\rangle + 2t\langle v,w\rangle + t^2\langle w,w\rangle$$
$$= \|v\|^2 + 2t\langle v,w\rangle + t^2\|w\|^2$$

Set $p(t) = at^2 + 2bt + c$ where
$a=\|w\|^2$, $b=\langle v,w\rangle$, $c=\|v\|^2$. So our ineq is

$$0 \le p(t).$$

Where is the max of $p$? $0=p'(t) = 2at+2b \Rightarrow t=-\frac{b}{a}$

So
$$0 \le p\left(-\frac{b}{a}\right) = a\left(-\frac{b}{a}\right)^2 + 2b\left(-\frac{b}{a}\right) + c$$
$$= \frac{b^2}{a} - 2\frac{b^2}{a} + c$$
$$= c - \frac{b^2}{a}$$
$$= \frac{ca-b^2}{a} = \frac{\|v\|^2\|w\|^2 - (\langle v,w\rangle)^2}{\|w\|}.$$

We conclude that $(\langle v,w\rangle)^2 \le \|v\|^2\|w\|^2$ and now take $\sqrt{\ }$'s.
Moreover, we have equality if and only if $v$ and $w$ are parallel.

We see that $-1 \le \dfrac{\langle v,w\rangle}{\|v\|\,\|w\|} \le 1$, so we can define angles $\theta$ between $v$ and $w$ by

$$\cos\theta = \frac{\langle v,w\rangle}{\|v\|\,\|w\|}.$$

(Use cosine since $\theta=0$ corresponds to being parallel)

Def'n Two vectors are orthogonal if $\langle v,w\rangle = 0$.

Example ①

$$v = \begin{pmatrix}1\\2\end{pmatrix},\ w = \begin{pmatrix}6\\-3\end{pmatrix} \text{ on } \mathbb{R}^2$$

$$\langle v,w\rangle = 1\cdot 6 + 2\cdot(-3) = 6-6 = 0$$

May not be orthog rel. to another inner product!

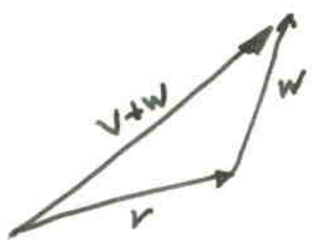
Example ② $p(x) = x$, $q(x) = x^2 - \frac{1}{2}$, $\langle p,q\rangle = \int_0^1 p(x)q(x)\,dx$

So $\langle x, x^2 - \frac{1}{2}\rangle = \int_0^1 x(x^2 - \frac{1}{2})\,dx = \cdots = 0$

Orthogonal polynomials!

Triangle Inequality

$\|v+w\| \le \|v\| + \|w\|$




Pf

$$\|v+w\|^2 = \langle v+w, v+w\rangle$$
$$= \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$$
$$= \|v\|^2 + 2\langle v,w\rangle + \|w\|^2 \le \|v\|^2 + 2\|v\|\,\|w\| + \|w\|^2$$
$$= (\|v\| + \|w\|)^2$$ Done.

What does this look like in certain cases?

$$\left|\sum_{i=1}^n v_i w_i\right| \le \sqrt{\sum_{i=1}^n v_i^2}\cdot\sqrt{\sum_{i=1}^n w_i^2}.$$

$\sqrt{\ \cdots}$ and for integrals.

**Theorem** Every inner product $\langle x,y\rangle$ on $\mathbb{R}^n$ is given by

$$\langle x,y\rangle = x^T K y$$

for some +ve definite $n\times n$ matrix $K$.

**Example**

$$K = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$$

$$\begin{aligned} q(x) = x^T K x &= 4x_1^2 - 4x_1x_2 + 3x_2^2 \\ &= (4x_1^2 - 4x_1x_2 + x_2^2) + 2x_2^2 \\ &= (2x_1 - x_2)^2 + 2x_2^2. \end{aligned}$$

Thus $q(x) > 0$ unless $x_2 = 0$, $2x_1 = x_2 \Rightarrow x_1 = 0$ ie $x = 0$.

**Example**

$$K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$$\begin{aligned} q(x) &= ax_1^2 + 2bx_1x_2 + cx_2^2 \\ &= a\left(x_1^2 + 2\frac{b}{a}x_1x_2 + \frac{b^2}{a^2}x_2^2\right) + cx_2^2 - \frac{b^2}{a}x_2^2 \\ &= a\left(x_1 + \frac{b}{a}x_2\right)^2 + \frac{ca-b^2}{a}x_2^2 \end{aligned}$$

This is $\geq 0$ iff $a > 0$ and $ca - b^2 > 0$

$\Rightarrow$ +ve (1,1) component AND +ve determinant

Can also define positive semi-definite (ie $q(x) = x^TAx \geq 0 \ \forall x$) negative definite and negative semi-definite for symmetric matrices and quadratic forms.

Null Directions!

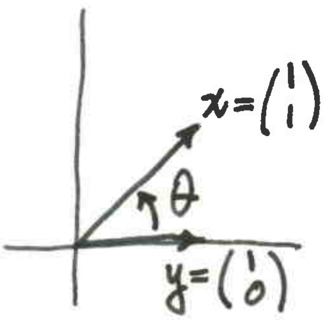
Now for a <u>warning</u>. Keep your wits about you. Consider the following inner product on $\mathbb{R}^2$:

$$\langle x, y \rangle = x^T \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} y$$

$$= (x_1 \;\; x_2) \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= (x_1 - x_2 \quad -x_1 + 2x_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= x_1 y_1 - x_2 y_1 - x_1 y_2 + 2 x_2 y_2 .$$




So what is the angle $\theta$ here? Well:

$$\langle x, y \rangle = (1 \;\; 1) \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (0 \;\; 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \times 1 + 1 \times 0$$

$$= 0$$

Thus $x$ and $y$ are orthogonal, and so $\theta = 90°$ !

Everything now is relative to some inner product!