Math 415 Lecture 15

Section 3.4.

Let $\langle x,y\rangle$ be an inner product on $\mathbb{R}^n$. How can we characterize it? Write

$$x = x_1 e_1 + \cdots + x_n e_n$$
$$y = y_1 e_1 + \cdots + y_n e_n.$$

Then
$$\langle x,y\rangle = \left\langle \sum x_i e_i , \sum y_j e_j \right\rangle = \sum_{i,j=1}^{n} x_i y_j \langle e_i, e_j\rangle.$$

Set $k_{ij} = \langle e_i, e_j\rangle$. Then

$$\langle x,y\rangle = x^T K y, \quad K = (k_{ij})$$

Why is K symmetric?
$$K^T = K$$

Conversely, if K is symmetric and we define $\langle x,y\rangle$ by $\langle x,y\rangle = x^T K y$, then

$$\langle x,y\rangle = \underbrace{x^T K y}_{\text{a scalar}} = (x^T K y)^T = y^T K^T x = y^T K x = \langle y,x\rangle$$

Also, this fn in linear in x and linear in y (Why?)

How about positivity:

$$\|x\|^2 = \langle x,x\rangle = x^T K x = \sum_{i,j=1}^{n} x_i x_j k_{ij} \equiv \overset{\text{quadratic form corresp to } K}{q(x)} \geq 0 \text{ for all } x \in \mathbb{R}^n$$
with equality iff $x = 0$.

Def'n An $n \times n$ symmetric matrix K is positive definite if
$x^T K x > 0$ for all $x \neq 0$ in $\mathbb{R}^n$.

Similarly we say its quadratic form $q(x)$ is +ve definite.

Def'n: Let V be a vector space and let $v_1, \dots, v_n$ be members of V. Then

$$K = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \cdots & \langle v_1, v_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle v_n, v_1\rangle & \langle v_n, v_2\rangle & \cdots & \langle v_n, v_n\rangle \end{pmatrix}$$

is call the <u>Gram matrix</u> corresponding to the inner product and the vectors $v_1, \dots, v_n$.

Why is an Gram matrix symmetric?

<u>Thm</u> All Gram matrices are positive semidefinite

Moreover they are +ve definite iff $v_1, \dots, v_n$ are lin independent

<u>Pf</u>

$$x^T K x = \sum_{i,j=1}^{n} x_i x_j \langle v_i, v_j\rangle$$

$$= \langle \sum_{i=1}^{n} x_i v_i , \sum_{j=1}^{n} x_j v_j \rangle$$

$$= \|w\|^2 \quad \text{where } w = \sum_{i=1}^{n} x_i v_i$$

≥ 0 Done.

Moreover, for equality we need $w = \sum_{i=1}^{n} x_i v_i = 0$. All of the x_i's are zero, i.e. $x = 0$, iff $v_1, \dots, v_n$ are linearly indep!

If we are using the Euclidean inner product on $\mathbb{R}^m$ $\langle v, w\rangle = \sum_{i=1}^{m} v_i w_i$ then the preceding K is simple to compute:

$$K = A^T A \quad \text{where } A = (v_1 \dots v_n)$$

$$= \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} (v_1 \dots v_n) = \begin{pmatrix} v_1 \cdot v_1 & \cdots & v_1 \cdot v_n \\ \vdots & & \\ v_n \cdot v_1 & \cdots & v_n \cdot v_n \end{pmatrix}$$

Example $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$ in $\mathbb{R}^3$

$$K = A^T A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & 45 \end{pmatrix}$$

Prop'n Given an $m \times n$ matrix A, the following are equiv

a) $K = A^T A$ is an $n \times n$ positive definite Gram matrix

b) A has linearly indep columns

c) rank $A = n \leq m$

d) $\ker A = \{0\}$. ← check!

Why is this true?

If $\langle v, w \rangle$ is non-Euclidean, then what? Then $\langle v, w \rangle = v^T C w$ for some +ve definite sym matrix C ($m \times m$)

So $k_{ij} = \langle v_i, v_j \rangle = v_i^T \underbrace{C v_j}_{j^{\text{th}} \text{ column of } CA}$

i^{th} row of A^T

So $K = A^T C A$.

Thm If A is $m \times n$ with linearly indep columns and C is $m \times n$ +ve definite then

$K = A^T C A$ is +ve definite sym (Gram) matrix $n \times n$.

Example Consider $V_1 = 1$, $V_2 = x$, $V_3 = x^2$ on $C^0[0,1]$.
Then

$$\langle v_1, v_1\rangle = \int_0^1 1\cdot 1\,dx = \|1\|^2 = 1$$
$$\langle v_1, v_2\rangle = \int_0^1 1\cdot x\,dx = \frac{1}{2}$$
$$\langle v_1, v_3\rangle = \int_0^1 1\cdot x^2\,dx = \frac{1}{3}$$

$$\langle v_2, v_1\rangle = \frac{1}{2}$$
$$\langle v_2, v_2\rangle = \int_0^1 x\cdot x\,dx = \frac{1}{3}$$
$$\langle v_2, v_3\rangle = \int_0^1 x\cdot x^2\,dx = \frac{1}{4}$$
$$\langle v_3, v_1\rangle = \frac{1}{3}$$
$$\langle v_3, v_2\rangle = \frac{1}{4}$$
$$\langle v_3, v_3\rangle = \int_0^1 x^2\cdot x^2\,dx = \frac{1}{5}$$

$$K = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}.$$

Example In $C^0[-\pi,\pi]$ consider $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$ and set $v_1 = 1$, $v_2 = \cos x$, $v_3 = \sin x$

$$\therefore \quad \langle v_1, v_1\rangle = \int_{-\pi}^{\pi} 1^2\,dx = 2\pi$$

$$\langle v_1, v_2\rangle = \int_{-\pi}^{\pi} 1\cdot\cos x\,dx = 0$$
$$\langle v_1, v_3\rangle = \int_{-\pi}^{\pi} 1\cdot\sin x\,dx = 0$$

etc.

$$\Rightarrow K = \begin{pmatrix} 2\pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}$$