Math 415 Lecture 16

Sec 3.5 Completing the square

Recall

$$q(x) = ax_1^2 + 2bx_1x_2 + cx_2^2$$
$$= a\left(x_1 + \frac{b}{a}x_2\right)^2 + \frac{ac-b^2}{a}x_2^2$$

Set $y_1 = x_1 + \frac{b}{a}x_2$, $y_2 = x_2$

$$\Rightarrow \quad q(x_1,x_2) = \hat{q}(y_1,y_2) = ay_1^2 + \frac{ac-b^2}{a}y_2^2$$

a sum of squares.

$$= y^TDy \ , \quad D = \begin{pmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{pmatrix}.$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + \frac{b}{a}x_2 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}}_{L^T} x .$$

Then
$$y^TDy = (L^Tx)^TD(L^Tx)$$
$$= x^TLDL^Tx$$
$$= x^TKx \qquad \text{where } K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

so

$$K = LDL^T$$ the specialized LDV reduction/factorization for symmetric matrices.

We can use this to "complete the square: set $y = L^Tx$.
Then $y^TDy = d_1y_1^2 + \cdots + d_ny_n^2 = x^TLDL^Tx = x^TKx = q(x)$.