Math 415 Lecture 18

Sec 4.2. Minimization of quadratic functions

ok if we are minimizing!

$$
\begin{aligned}
p(x) &= \|Ax-b\|^2 \\
&= \langle Ax-b, Ax-b\rangle \\
&= \langle Ax, Ax\rangle - 2\langle Ax, b\rangle + \langle b, b\rangle \\
&= \Big\langle \sum_{i=1}^n x_i v_i, \sum_{i=1}^n x_i v_i \Big\rangle - 2\Big\langle \sum_{i=1}^n x_i v_i, b\Big\rangle + \|b\|^2 \\
&= \sum_{i,j=1}^n x_i x_j k_{ij} - 2\sum_{i=1}^n f_i x_i + C
\end{aligned}
$$

where $k_{ij} = \langle v_i, v_j\rangle$, $K = (k_{ij})$, $f_i = \langle v_i, b\rangle$, $f = (f_i)$, $c = \|b\|^2$.

How can we minimize $p$? One way is to use calculus. The other way is to complete the square:

$$p(x) = x^T K x - 2x^T f + c$$

Let $x^*$ be the unique soln of $Kx = f$, call it $x^*$ — Why is it unique? Then $Kx^* = f$ or $x^* = K^{-1}f$. Then

$$
\begin{aligned}
p(x^*) = p(K^{-1}f) &= c + f^T K^{-T} \cancel{K K^{-1}} f - 2 f^T K^{-T} f \\
&= c - f^T K^{-T} f \\
&= c - (Kx^*)^T K^{-T} K x^* \\
&= c - x^{*T} \cancel{K^T K^{-T}} K x^* \\
&= c - x^{*T} K x^*.
\end{aligned}
$$

Note that
$$
\begin{aligned}
p(x) &= x^T K x - 2x^T K x^* + c \\
&= (x - x^*)^T K (x - x^*) + (c - x^{*T} K x^*)
\end{aligned}
$$

Since $K$ is +ve definite we see that $x^*$ is the minimum of $p(x)$.

<u>Example</u> Minimize

$$p(x,y,z) = x^2 + 2xy + xz + 2y^2 + yz + 2z^2 + 6y - 7z + 5.$$

Write as $p(x,y,z) = \vec{x}^T K \vec{x} \cancel{+} -2\,\vec{x}\cdot f + c$

where $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $K = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix}$, $c = 5$, $f = \begin{pmatrix} 0 \\ -3 \\ +\frac{7}{2} \end{pmatrix}$

$$K = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}^T}_{LDL^T \text{ factorization of } K.}$$

$D$ has only positive entries, so $K$ is +ve definite

Reminder set $\vec{y} = L^T \vec{x} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. Then

$$\vec{y}^T D \vec{y} = (L^T \vec{x})^T D L^T \vec{x} = \vec{x}^T L D L^T \vec{x} = \vec{x}^T K \vec{x}$$

$\vec{y}^T D \vec{y} = y_1^2 + y_2^2 + \frac{7}{4} y_3^2$. So $\vec{x}^T K \vec{x} \geq 0$. And $\vec{x}^T K \vec{x} = 0 \Rightarrow$
$\vec{y}^T D \vec{y} = 0 \Rightarrow y_1 = y_2 = y_3 = 0 \Rightarrow \vec{y} = 0 \Rightarrow \vec{x} = L^{-T} \vec{y} = 0$, so +ve definite

Thus by our thm, $p$ is minimized by the sol'n $\vec{x}^*$ of $Kx = f$
Use forward + back substitution on

$$U\vec{x}^* = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{4} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{c}$$

$$L\vec{c} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = f = \begin{pmatrix} 0 \\ -3 \\ 7/2 \end{pmatrix}.$$

$$\Rightarrow \quad \vec{x}^* = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \text{ and } \min = p(\vec{x}^*) = p(2,-3,2) = -11$$

Theorem: If $K$ is +ve definite, then $p(x) = x^T K x - 2x^T f + C$ has a unique global minimum at $x^*$ satisfying $Kx^* = f$. If $K$ is only +ve semi-definite, then every solution of $Kx^* = f$ gives a global min of $p$, but it is not unique since $p(x^* + z) = p(x^*)$ for any $z \in \ker K$. In all other cases $p(x)$ has no global minimum.

$$\begin{aligned} p(x^*+z) &= \langle x^*+z, x^*+z \rangle - 2(x^*+z)^T f + c \\ &= \langle x^*, x^* \rangle + 2\langle x^*, z \rangle + \langle z, z \rangle - 2(x^*+z)^T K x^* + C \\ &= x^{*T} K x^* + 2x^{*T} \underbrace{Kz}_{0} + z^T \underbrace{Kz}_{0} - 2x^{*T}(\underbrace{Kx^*}_{f} + \underbrace{Kz}_{0}) + C \\ &= x^{*T} K x^* - 2x^{*T} f + C = p(x^*). \end{aligned}$$

Rest of proof? ....