Math 415 Lecture 19

Sec 4.3 The closest point problem:

Given a point $b \in \mathbb{R}^m$ and a subset V of $\mathbb{R}^m$, find the point $\hat{v}$ on V closest to b. Set

$$d(v, b) = \|v - b\| = \text{distance between } v \text{ and } b.$$

Suppose V is a subspace and we can find a basis $v_1, \dots, v_n$. Then

$$v = x_1 v_1 + \dots + x_n v_n = Ax \quad \text{where } A = \dots, \; x = \dots$$

is our representn/charactn of each point on V. That is, V is $\operatorname{rng} A$. Then the problem is

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; \|Ax - b\|$$

If min is zero, then b lies on V, otherwise we get the closest point.

Minimizing $p(x)$:

$$\frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = 0 \quad \Rightarrow \quad x = x^*$$

$$\left.\frac{\partial^2 p}{\partial x_1^2}\right|_{x^*} > 0 , \qquad \underbrace{\left.\frac{\partial^2 p}{\partial x_1^2}\right|_{x^*} \left.\frac{\partial^2 p}{\partial x_2^2}\right|_{x^*} - \left(\left.\frac{\partial^2 p}{\partial x_1 \partial x_2}\right|_{x^*}\right)^2}_{\text{descriminant}} > 0$$

$$H = \begin{pmatrix} \left.\frac{\partial^2 p}{\partial x_1^2}\right|_{x^*} & \left.\frac{\partial^2 p}{\partial x_1 \partial x_2}\right|_{x^*} \\ \left.\frac{\partial^2 p}{\partial x_1 \partial x_2}\right|_{x^*} & \left.\frac{\partial^2 p}{\partial x_2^2}\right|_{x^*} \end{pmatrix}$$ = Hessianmatrix, must be +ve definite for a local minimum.

Review Example 4.6

Example Based Upon Remark on pg 192. $C^0[0,1]$

Find the function in $V = \text{span}\{\underset{v_1}{1}, \underset{v_2}{x}\}$ that is closest to $x^2 \leftarrow b$

$$p(y) = p(y_1, y_2) = \| y_1 v_1 + y_2 v_2 - b \|^2$$
$$= \langle \Sigma y_i v_i , \Sigma y_j v_j \rangle - 2 \langle \Sigma y_i v_i , b \rangle + \langle b , b \rangle$$
$$= \sum_{i,j=1}^{2} y_i y_j \underbrace{\langle v_i , v_j \rangle}_{k_{ij}} - 2 \sum_{i=1}^{2} y_i \underbrace{\langle v_i , b \rangle}_{f} + \underbrace{\|b\|^2}_{c}$$

$k_{11} = \int_0^1 1 \cdot 1 \, dx = 1$, $k_{12} = \int_0^1 1 \cdot x \, dx = \frac{1}{2} = k_{21}$, $k_{22} = \int_0^1 x \cdot x \, dx = \frac{1}{3}$

$f_1 = \int_0^1 1 \cdot x^2 dx = \frac{1}{3}$, $f_2 = \int_0^1 x \cdot x^2 dx = \frac{1}{4}$, $c = \int_0^1 x^2 x^2 dx = \frac{1}{5}$

$K = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$, $f = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \end{pmatrix}$, $c = \frac{1}{5}$. Apply the theorem! Is K +ve definite?

$$Kx^* = f \Rightarrow x^* = K^{-1}f = \frac{1}{1\cdot\frac{1}{3}-\frac{1}{4}}\begin{pmatrix}\frac{1}{3} & -\frac{1}{2}\\ -\frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}\frac{1}{3}\\ \frac{1}{4}\end{pmatrix} = \begin{pmatrix}-\frac{1}{6}\\ 1\end{pmatrix}$$

$\Rightarrow$ $x - \frac{1}{6}$ is the fn closest to x^2 in $C^0[0,1]$ in the span of $\{1, x\}$.

Least Squares Approximation

$$Ax = b$$

Suppose this doesn't have a sol'n. What is the closest to satisfy it?

Def'n A least squares sol'n to

$$Ax = b$$

is a vector $x^* \in \mathbb{R}^n$ that minimizes $\|Ax - b\|$.

But the vectors Ax for $x \in \mathbb{R}^n$ form the range of A, so if we set $V = \operatorname{rng} A$, then the least squares sol'n to $Ax = b$ is equivalent to finding the closes point in $\operatorname{rng} A$ to b.

Review Example 4.9

Thm Assume $\ker A = \{0\}$. Set $K = A^TA$ and $f = A^Tb$. Then let x^* be the unique solution of

$$Kx = f \text{ (normal eqn's) or } A^TAx = A^Tb$$

ie $x^* = (A^TA)^{-1}A^Tb$

Then the least squares error is

$$\|Ax^* - b\|^2 = \|b\|^2 - f^Tx^* = \|b\|^2 - b^TA(A^TA)^{-1}A^Tb.$$